NONLINEAR RESPONSE OF SHELLS TO BLAST AND IMPACT

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1 Introduction
This study is concerned with the transient response of shells to blast and impact loads. A general approach is developed to derive the equations of motion, boundary conditions and approximate solutions for plates and shells with various geometries. The formulation includes the effects of large deflections and initial geometric imperfections. Laminated composite and sandwich structures can be analyzed using this approach. The response to pressure pulses typical of blast loads is examined using a simple model. Results are in good agreement with published results. Geometrical nonlinearities and initial imperfections are found to have significant effects on the response. Composite structures can experience significant damage when subjected to foreign object impacts. With cylindrical shells several examples show that often several of the lowest modes have very close frequencies and many modes participate in the response. In those cases, the Donnell-Mushtari-Vlasov theory was shown not to be sufficiently accurate.

2 Problem formulation
A general formulation is developed to analyze the dynamic behavior of shells made of composite materials including functionally graded materials. The model can handle different geometries, different kinematic assumptions and complicating factors such as large deflections and initial imperfections. First we examine how to describe the kinematics of the deformation and then how to derive the equations of motion and approximate solution.

2.1 Kinematics of the deformation
A curvilinear system is used to describe the motion in a general way that can be specialized later to shells of different geometries: cylindrical, spherical, conical and others. Starting with the three dimensional Green-Lagrange strain tensor, several assumptions and simplifications are made to express the strains in terms of the displacements of the reference surface.

An orthogonal curvilinear coordinate system is used where the \( \alpha \) and \( \beta \) coordinates define the reference surface and \( \zeta \) is in the normal direction. The Green-Lagrange strain-displacement relations (Eqs. A1, A2) in such a system given by Saada [1] are listed below for completeness.

The 1850 Kirchhoff plate theory is based on two simple assumptions: “(1) Normals to the undeformed middle surface remain normal to the deformed middle surface without change in length; (2) Transverse normal stress may be neglected in the in-plane stress strain relation” [2]. Love’s 1888 shell theory is also based on those first two assumptions and in addition, it is also assumed that “(3) The shell is so thin that \( \zeta/R \) terms may be omitted compared to unity” [2]. In other words, the thickness is small compared to the two radii of curvature. Love’s thin shell assumption is adopted here so \( 1 + \zeta/R \) terms are removed from Eq. A2.

Since the introduction of Kirchhoff-Love theory, many models based on other kinematic assumptions. For example, in the first order shear deformation theory (FSDT), normals to the undeformed middle surface remain straight and do not change length but are no longer normal to the deformed middle surface (Eqs. A3). Substituting into Eqs. (A1, A2), the in-plane strains \( \{\varepsilon\} = [\varepsilon_{\alpha\alpha}, \varepsilon_{\beta\beta}, \varepsilon_{\alpha\beta}]^T \) can be written as

\[
\{\varepsilon\} = \{\varepsilon^{sh}\} + \zeta \{\kappa\} + \{\varepsilon^{nl}\} \quad (1)
\]
where the midplane strains \( \{ \varepsilon^o \} \) and the curvatures \( \{ \kappa \} \) are given in the appendix (Eq. A4, A5). The linear components of the transverse shear strains are given in Eq. (A6). These linear strains and curvatures can be found in many publications (e.g. [3-5]).

Expressions for the nonlinear stresses contain many terms. For the FSDT, terms with the following powers of the transverse coordinate are present: \( \varepsilon^o, \varepsilon^1, \varepsilon^2 \). While a few investigators retain all terms, often, only the first term is retained. Then, nonlinearities affect only the midplane strains, not the bending strains. With the FSDT, the nonlinear strain is then given by Eq. A7. Further simplification is possible by considering that nonlinear effects are due to large transverse deflections only. This results in the Sanders nonlinear strains. Since Love’s thin shell assumption was made, it is logical to assume that \( w/R \) terms are also small if the transverse displacements remain small compared to the radius of curvature. This yields the von Karman strains (Eqs. A8). That approach can be used to determine the nonlinear components of the transverse shear strains. For the FSDT, using von Karman’s approach these become zero. This general approach can be used in conjunction with any assumptions made about the kinematics of the deformation. This includes the Kirchhoff-Love assumption, Reddy’s third order theory and others.

To account for initial geometrical imperfections, it is assumed that the total distance of a point from the idealized reference surface is the sum of the initial imperfection \( \tilde{w} \) plus \( w \), the transverse displacement caused by the mechanical loading. Therefore, \( w \) should be replaced by \( w + \tilde{w} \) in Eqs. A9. However, since the initial configuration is assumed to be stress free, \( \tilde{w} \) should not appear in the linear strains and curvatures. Similarly, the nonlinear strain components should not include \( (\partial \tilde{w}/\partial \alpha)^2 \) and \( (\partial \tilde{w}/\partial \beta)^2 \) terms. Consequently, the nonlinear strains are given by Eqs. A8 when initial imperfections are included. This brief overview shows how many formulations in the literature can be recovered.

### 2.2 Governing equations

The strain energy \( U \), the work done by the external forces \( W \), and kinetic energy \( K \) are evaluated as shown in Appendix B. The equations of motion and boundary conditions are derived using Hamilton’s principle

\[
\delta \int_{t_i}^{t_f} [K - (U + V)] dt = 0 \tag{2}
\]

It is generally possible to obtain exact solutions to the equations of motion. Approximate solutions are obtained using Lagrange’s equations

\[
\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{a}_j} \right) + \frac{\partial U}{\partial a_j} = \frac{\partial W}{\partial a_j} \tag{3}
\]

when the displacements are expressed in terms of approximation functions \( \phi_{ij} \) and pseudo-displacements \( a_j \).

### 3 Plates with Initial Imperfections

Considering the case of shells with zero curvatures a simple and yet accurate one degree of freedom model is developed. This model is used to study the dynamic response to various pressure pulses and examine the effects of nonlinearities and initial deflections.

#### 3.1 Governing equations

For certain plate geometries, \( \alpha = x \) and \( \beta = y \) the Lame constants \( A = B = 1 \), and \( 1/R_x = 1/R_y = 0 \). Neglecting shear deformations, the rotations of line segments initially normal to the midsurface are related to the transverse displacement (Eq. B1). The curvatures and nonlinear strains are given by Eqs. (B2, B3).

Simple models based on a one term approximation have been presented. Dowell and Ventres [6] report that for isotropic, rectangular, simply supported plates several previous investigators obtained the same model assuming that

\[
w = a_1 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{4}
\]
and neglecting the in-plane displacements \( u \) and \( v \). This displacement approximation is the first mode shape for the linear plate. Ref 6 suggests that in addition, the in-plane displacements could be taken as

\[
\begin{align*}
u &= b_{22} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} \\v &= c_{12} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b}
\end{align*}
\]

Eqs. (5,6) satisfy the boundary conditions for immovable edges. The kinetic energy of the plate is taken to depend only on the transverse displacement \( w \). Introducing those in-plane displacement was shown not to affect the results significantly. Singh et al [7] studied rectangular plates with the displacement approximation given by Eqs. (4-6)

Louca et al [8, 9] studied simply supported plates with initial geometrical imperfections \( \tilde{w} \), and a single-mode approximation for transverse displacement \( w \) given by Eq (3). A global imperfection of the form

\[
\tilde{w} = a_{0} \frac{\pi x}{a} \sin \frac{\pi y}{b}
\]

is considered. Louca et al [10] consider isotropic plates with imperfections including the in-plane displacements \( u \) and \( v \) as in Eqs. (4,5).

For symmetric balanced laminates, the strain energy is evaluated using Eq. (B4) in terms of the in-plane and bending rigidities. The kinetic energy and the work of the external forces are obtained as indicated in Appendix B. Then, Eq. 3 yields the equation of motion

\[
m\ddot{a}_{1} + k_{NL}(a_{0}^{2} + 2a_{0}\dot{a}_{0})(a_{1} + a_{0}) = F_{0} f(t)
\]

where \( m = \rho ah/4 \) is the effective mass of the plate, \( r = a/b \), and the linear bending rigidity \( k \) and is the nonlinear rigidity \( k_{NL} \) are given by

\[
k = \frac{\pi^{3} b}{128a^{4}} \left[ \frac{9A_{1} + 2(A_{2} + 2A_{6})r^{2} + 9A_{22}r^{4}}{A_{12} + 2A_{66}r^{2} + A_{22}r^{4}} \right]
\]

For a distributed pressure \( p_{0} f(t) \), \( F = 4p_{0} ab/\pi^{2} \). For isotropic plates, the equation derived by Louca and Harding [8] is recovered. Duffing’s equation with a cubic nonlinear term is recovered when \( a_{0} = 0 \). Geometrical imperfections introduce terms proportional to \( a_{1}^{2}a_{0} \) and \( a_{0}^{2} \) that affect the dynamics.

The natural frequencies of an isotropic, simply supported, square plate with a side to thickness ratio \( a/h = 120 \) and \( \nu = 0.3 \) obtained by the present approach are in excellent agreement with those from [11] as are the static deflections under a uniform pressure (Fig. 1).

This approach can be generalized to obtain one term approximations for other support conditions along the edges. It can also be used to obtain higher order models.

### 3.2 Blast loading on composite plates

For rectangular plates, the response to a uniform pressure pulse is dominated by the first mode and the contribution of other modes can be neglected [12]. This result applies to other systems with clearly separated frequencies and a single mode analyses are appropriate in those cases. Starting with an example to demonstrate the effect of nonlinearities, a non-dimensional form of the one degree of freedom model is used to study the response of plates to a step pressure, to rectangular and exponential pressure pulses.

The response of rectangular simply supported plates subjected to a uniformly distributed step pressure load has been studied extensively. Here we consider an example first presented by Akay [13] and used by many others including Chen and Sun [14] where \( a = b = 2.438 \text{ m}, h = 6.35 \text{ mm}, \nu = 0.25, E = 68.9 \text{ GPa} \) and \( \rho = 2547 \text{ kg/m}^{3} \) and \( p_{0} = 4784 \text{ N/m}^{2} \). Results from the one term approximation (Fig. 2) are in excellent agreement with those obtained in [13, 14]. Nonlinear effects are significant and must be accounted for. Because of the nonlinear stiffening,
the amplitude of the response is smaller and the period of the oscillations is shorter.

Neglecting initial imperfections in the one-mode approximation, the dynamic response is governed by the non-dimensional Duffing equation

\[ x'' + x + \varepsilon x^3 = f(\tau) \quad (9) \]

where \( x = a_1 k / F \) is the non-dimensional displacement, \( \tau = t \sqrt{k / m} \), and \( \varepsilon = k_{nl} F^2 / k^3 \).

For a step pressure load (\( f(t) = H(t) \)), the first integral of Eq. 9

\[ x^2 + x^2 + \frac{\varepsilon}{2} x^4 - x = 0 \quad (10) \]

is used to calculate \( x_{max} \), the maximum value of \( x \). The velocity reaches a maximum \( x'_{max} \) determined from Eq. 10 and the corresponding displacement \( x_o \) is found solving \( x + \varepsilon x^3 = 0 \). Table 1 shows that as \( \varepsilon \) increases \( x_{max} \), \( x_o \) and \( x'_{max} \) decrease.

For a linear system, the phase diagram is a unit circle centered at (1,0). For nonlinear systems the shape of the trajectory differs significantly different from a circle (Fig. 3). For small values of \( \varepsilon \) and \( x \) remains small and, neglecting the second and third terms on the left hand side of Eq. , the response is impulsive. That is,

\[ x'(\tau) = \int_0^\tau f(\tau) d\tau^* \]

and the response depends on the area under the \( f(\tau) \) versus \( \tau \) curve and not the shape of the curve. For a step load, the impulsive response is given by \( x = \tau^2 / 2 \) and \( x' = \tau \). An approximate solution of the form \( x = A(1 - \cos \omega t) \) was also obtained where the amplitude and the frequency are obtained from

\[ A + \frac{5}{2} \varepsilon A^3 = 1 \quad \text{and} \quad \omega^2 = 1 + \frac{15}{4} \varepsilon A^2 \].

Initially, the response is impulsive and the simple approximation given here is adequate even when \( \varepsilon \) is large (Fig. 4).

For a rectangular pulse with a finite duration, the phase diagram starts with a circle A centered at (0,1) during the loading phase and then switches to a circle B centered at (0,0) during the free vibration phase (Fig. 4). The response to a rectangular pulse \( f(\tau) = 1 \) when \( 1 \leq \tau \leq 2 \) shows a similar behavior for the nonlinear case. The response is impulsive when the pulse duration is so small that the effect of the restoring forces \( x + \varepsilon x^3 \) terms in Eq. 9) are negligible and the parabola is a good approximation to the red circle.

Including the effects of initial imperfections, the one-mode approximation is non-dimensionalized as

\[ x'' + x + \varepsilon (x^3 + 3x^2 x_o + 2x x_o^2) = f(\tau) \quad (12) \]

where \( x \) and \( \varepsilon \) are defined as before and \( x_o = a_k k / F \).

With \( f(\tau) = 0 \) and an initial displacement \( a_1(0) \), the new variables are defined as \( x = a_1 / a_1(0) \), \( x_o = a_o / a_1(0) \) and \( \varepsilon = a_1^2(0) k_{nl} / k \). The restoring force \( F_R = x + \varepsilon (x^3 + 3x^2 x_o + 2x x_o^2) \) is strongly nonlinear (Fig. 5). The imperfections have a strong effect on the dynamics of the system (Fig. 6).

The dynamic response to blast loading has been studied by a number of investigators including Kazanci [15-17]. The linear response to an exponential blast governed by the non-dimensional equation of motion \( x'' + x = e^{-\tau/t_o} \) is

\[ x = \left( e^{-\tau/t_o} + \frac{1}{t_o} \sin\tau - \cos\tau \right) \left( 1 + \frac{1}{\tau_o} \right) \quad (13) \]

The response consists of a quasi-static exponential part and an oscillating part (Fig. 7). As \( \tau \) becomes large, the loading has died down and the radius \( R \) of the circle for the free vibration phase (Fig. 8) is given by

\[ R^2 = x^2 + \dot{x}^2 \approx 1/ \left( 1 + \frac{1}{\tau_o} \right) \quad (14) \]

When \( \tau_o \) is much larger than one, \( R \) is close to one. When \( \tau_o = 10 \), the motion settled on a circle centered at the origin with a radius very close to one (Fig. 8). As \( \tau_o \) becomes small, \( R \) becomes small and less energy is absorbed (Fig. 9). Figs. (8, 9) show that when \( \tau_o \) is large the maximum deflection occurs
during the loading phase while for small values of \( \tau \), this maximum occurs during the free vibration phase.

### 3.4 Impact on composite plates

The effects of foreign object impacts on composite structures have been studied extensively [18-20]. Using a single mode a nonlinear model for the plate and Hertz’s contact law for the local indentation between the plate and the projectile, the dynamics of the impact event is governed by

\[
\ddot{\ddot{x}} + \dddot{x} + k_{nn} \dot{x}^3 = k_c (y - \ddot{x})^{3/2} \quad (15)
\]

\[
\ddot{y} = -m \frac{m}{m_p} \dddot{y} (y - \ddot{y})^{3/2} \quad (16)
\]

where \( \tau = t \sqrt{k/m} \) is the non-dimensional time. The displacement of the plate and that of the projectile are non-dimensionalized as \( \ddot{x} = x \sqrt{k/m} / V \) and \( \ddot{y} = y \sqrt{k/m} / V \). In addition three non-dimensional parameters are introduced: the nonlinear stiffness of the plate \( k_{nn} = k_p m V^2 / k^2 \), the contact stiffness \( k_c = k y^{3/2} m^{1/2} / k^{3/4} \). \( k_p \) is the contact force, \( y \) is the displacement of the projectile, \( k_c \) is the contact stiffness and \( m_p \) is the mass of the projectile. A parametric study of this problem was presented in 2001 for linear plates \( (k_{nn} = 0) \) [19]. Consider cases in which \( m_p > m \) and the response is a global motion of the plate that can be captured by a one term approximation

The effect of the mass ratio on the contact force history is shown in Fig. 10. As \( m_p/m \) increases from 2 to 30 the maximum force and the impact duration increase. The results indicate that at the end of the impact, the two bodies separate with the plate and the projectile moving in the same direction. The velocity of the plate is greater than that of the projectile which retained a significant portion of its initial velocity. For this type of impact, the rigidity of the plate has a negligible effect on the contact force history.

### 4 Cylindrical shells

The dynamics of cylindrical shells is discussed in many publications [3-5, 21]. Leissa [21] provides an overview of the early studies that consider geometrically nonlinear effects. More recently, Amabili [22,23] come loopared several of the major theories.

Soedel [3] extended the Donnell-Mushtari-Vlasov theory to laminated orthotropic cylindrical shells. The motion is governed by two equations (Eqs. C1, C2) in terms of the transverse displacement \( w \) and a stress function \( \phi \).

One example in Loy et al [24] is a closed, simply supported, cylindrical shell with radius \( R = 1 \) m, length \( L = 20 \) m and thickness \( h = 2 \) mm. The shell is made of stainless steel with \( E = 207.788 \) GPa, \( \nu = 0.317756 \) and \( \rho = 8166 \) kg/m\(^3\). The natural frequencies \( \omega_{mm} \) where \( m \) is the number of half waves in the axial direction and \( n \) is the number of half waves in the radial directions, are calculated. This example is used as a benchmark problem in many publications.

Table 2 gives the first ten frequencies for \( m = 1 \). Frequencies denoted by \( \omega^5 \) are obtained using Soedel’s approach (Eq. C3) and \( \omega^1 \) refers to those given by Loy et al [24]. In Eq. C3, only \( \alpha_{11} \) accounts for the bending rigidities of the plate. The frequencies denoted by \( \omega_{app}^5 \) in Table 3 are obtained neglecting the term \( \alpha_{12} \alpha_{21} / \alpha_{22} \) in Soedel’s solution to highlight the effect of in-plane rigidities. These results show that \( \alpha_{12} \alpha_{21} / \alpha_{22} \) only affects the first few modes. Results in the last two columns of Table 3 are in good agreement except for the first few modes.

Both references [3,24] used Love’s thin shell with results in three equations of motion in terms of the three displacements of the mid surface: \( u_o, v_o \), and \( w \). Loy et al attack these equations directly using a numerical approach. On the other hand, an additional simplification is made in [3]: inertia terms associated with the displacements \( u_o, v_o \), are neglected. Only the inertia term associated with the transverse displacement \( w \) is retained in what is usually referred to as the Donnell-Mushtari-Vlasov theory. Results for orthotropic shells are presented in [3] and similar discrepancies between results obtained analytical results and finite element results is attributed to the fact that these inertia terms were neglected.
Further simplification can be achieved for long shells. Then, $\alpha_{11} \approx D_2 z(n/R)^{\frac{1}{2}}$ and frequencies calculated using $\omega^2 = D_2 z(n/R)^{\frac{1}{2}} (\rho h)$ are in very close agreement with the approximate solution in Table 3. Fig. 11 shows that for this shell, the effects of inplane motion are significant for small values on $n$. For higher values of $n$, all curves collapse on a single curve. Then the shell behaves like a ring of unit length with frequencies $\omega_R$.

Fig. 12 shows that the effects of inplane motion are less important when the thickness and length of the shell are increased.

**Conclusion**

This article briefly described a framework for the analysis of shells including the effects on geometric nonlinearities and imperfections. Shells with various geometries can be analyzed and the validity of various assumptions can be tested. Results for the analysis of plates and shells subjected to blast loadings and impacts indicate that geometric nonlinearities and initial imperfections can be significant. For shells, it was shown that the validity of some of the simpler theories might not be sufficient.

**Appendix A: Strain-displacement relations**

Nonlinear strain-displacement relations for a 3D body in orthogonal curvilinear coordinates

$$e_{aa} = e_{aa} + \frac{1}{2} (e_{a}^2 + e_{\beta a} e_{\alpha a} + e_{\alpha a}^2)$$

$$e_{\beta \beta} = e_{\beta \beta} + \frac{1}{2} (e_{\beta}^2 + e_{\beta \beta} e_{\alpha \beta} + e_{\alpha \beta}^2)$$

$$e_{\gamma \gamma} = e_{\gamma \gamma} + \frac{1}{2} (e_{\gamma}^2 + e_{\gamma \gamma} e_{\alpha \gamma} + e_{\alpha \gamma}^2)$$

$$\gamma_{a\beta} = \gamma_{a\beta} + e_{a\beta} + e_{aa} e_{\alpha \beta} + e_{\beta a} e_{\alpha \beta} + e_{\alpha \beta}^2 e_{\gamma \gamma}$$

$$\gamma_{a\gamma} = \gamma_{a\gamma} + e_{a\gamma} + e_{aa} e_{\alpha \gamma} + e_{\alpha \gamma}^2 e_{\gamma \gamma} e_{aa} e_{\alpha \beta} + e_{\alpha \beta} e_{\alpha \gamma} e_{\alpha \gamma} e_{aa} e_{\alpha \beta}$$

$$\gamma_{\beta \gamma} = \gamma_{\beta \gamma} + e_{\beta \gamma} + e_{\beta \gamma} e_{\alpha \beta} + e_{\alpha \beta} e_{\beta \gamma} + e_{\alpha \beta} e_{\alpha \gamma} e_{\alpha \gamma} e_{\beta \gamma} + e_{\alpha \beta} e_{\alpha \gamma} e_{\alpha \gamma} e_{\beta \gamma}$$

A and B being the Lame coefficients and $R_a$ and $R_\beta$ the principal radii of curvature.

$$e_{aa} = \frac{1}{(1 + \zeta / R_a)} \left( \frac{1}{\partial u}{\partial a} + \frac{v}{AB}{\partial A}{\partial R_a} + \frac{w}{R_a} \right)$$

$$e_{\beta \beta} = \frac{1}{(1 + \zeta / R_\beta)} \left( \frac{1}{\partial v}{\partial \beta} + \frac{u}{AB}{\partial B}{\partial R_\beta} + \frac{w}{R_\beta} \right)$$

$$e_{\gamma \gamma} = \frac{1}{(1 + \zeta / R_a)} \left( \frac{1}{\partial u}{\partial \gamma} + \frac{u}{AB}{\partial B}{\partial R_a} + \frac{v}{R_a} \right)$$

$$e_{\alpha \beta} = \frac{1}{(1 + \zeta / R_\beta)} \left( \frac{1}{\partial v}{\partial \alpha} + \frac{v}{AB}{\partial A}{\partial R_\beta} + \frac{u}{R_\beta} \right)$$

$$e_{\alpha \gamma} = \frac{1}{(1 + \zeta / R_a)} \left( \frac{1}{\partial v}{\partial \gamma} + \frac{v}{AB}{\partial B}{\partial R_a} + \frac{w}{R_a} \right)$$

$$e_{\beta \gamma} = \frac{1}{(1 + \zeta / R_\beta)} \left( \frac{1}{\partial u}{\partial \beta} + \frac{u}{AB}{\partial A}{\partial R_\beta} + \frac{v}{R_\beta} \right)$$

$$e_{\gamma \alpha} = \frac{1}{(1 + \zeta / R_a)} \left( \frac{1}{\partial v}{\partial \gamma} + \frac{v}{AB}{\partial B}{\partial R_a} + \frac{w}{R_a} \right)$$

$$e_{\gamma \beta} = \frac{1}{(1 + \zeta / R_\beta)} \left( \frac{1}{\partial u}{\partial \gamma} + \frac{u}{AB}{\partial A}{\partial R_\beta} + \frac{v}{R_\beta} \right)$$

$$e_{\gamma \alpha} = \frac{1}{(1 + \zeta / R_a)} \left( \frac{1}{\partial u}{\partial \gamma} + \frac{u}{AB}{\partial A}{\partial R_a} + \frac{w}{R_a} \right)$$

$$e_{\gamma \beta} = \frac{1}{(1 + \zeta / R_\beta)} \left( \frac{1}{\partial v}{\partial \gamma} + \frac{v}{AB}{\partial B}{\partial R_\beta} + \frac{w}{R_\beta} \right)$$

$$e_{\gamma \alpha} = \frac{1}{(1 + \zeta / R_a)} \left( \frac{1}{\partial v}{\partial \gamma} + \frac{v}{AB}{\partial B}{\partial R_a} + \frac{w}{R_a} \right)$$

$$e_{\gamma \beta} = \frac{1}{(1 + \zeta / R_\beta)} \left( \frac{1}{\partial u}{\partial \gamma} + \frac{u}{AB}{\partial A}{\partial R_\beta} + \frac{w}{R_\beta} \right)$$

$$e_{\gamma \alpha} = \frac{1}{(1 + \zeta / R_a)} \left( \frac{1}{\partial v}{\partial \gamma} + \frac{v}{AB}{\partial B}{\partial R_a} + \frac{w}{R_a} \right)$$

$$e_{\gamma \beta} = \frac{1}{(1 + \zeta / R_\beta)} \left( \frac{1}{\partial u}{\partial \gamma} + \frac{u}{AB}{\partial A}{\partial R_\beta} + \frac{w}{R_\beta} \right)$$

$$e_{\gamma \alpha} = \frac{1}{(1 + \zeta / R_a)} \left( \frac{1}{\partial v}{\partial \gamma} + \frac{v}{AB}{\partial B}{\partial R_a} + \frac{w}{R_a} \right)$$

$$e_{\gamma \beta} = \frac{1}{(1 + \zeta / R_\beta)} \left( \frac{1}{\partial u}{\partial \gamma} + \frac{u}{AB}{\partial A}{\partial R_\beta} + \frac{w}{R_\beta} \right)$$

With the FSDT, the deformation is described by

$$u(\alpha, \beta, \zeta) = u^0(\alpha, \beta) + \zeta \psi_a(\alpha, \beta)$$

$$v(\alpha, \beta, \zeta) = v^0(\alpha, \beta) + \zeta \psi_\beta(\alpha, \beta)$$

$$w(\alpha, \beta, \zeta) = w^0(\alpha, \beta)$$

Substitution into Eqs. A2 gives the linear midplane strains, the curvatures, and the transverse shear strains as

$$\{e\} = \begin{bmatrix} e_{aa}^o \\ e_{\beta \beta}^o \\ e_{\gamma \gamma}^o \\ e_{\alpha \beta}^o \end{bmatrix} = \begin{bmatrix} \frac{1}{A \partial A}{\partial a} + \frac{v}{AB}{\partial A}{\partial R_a} + \frac{w}{R_a} \\ \frac{1}{B \partial B}{\partial \beta} + \frac{u}{AB}{\partial B}{\partial R_\beta} + \frac{w}{R_\beta} \\ \frac{1}{A \partial A}{\partial \gamma} + \frac{v}{AB}{\partial B}{\partial R_a} + \frac{u}{R_a} \\ \frac{1}{B \partial B}{\partial \alpha} + \frac{u}{AB}{\partial B}{\partial R_\beta} + \frac{v}{R_\beta} \end{bmatrix}$$

$$\{\kappa\} = \begin{bmatrix} \kappa_{aa} \\ \kappa_{\beta \beta} \\ \kappa_{\gamma \gamma} \\ \kappa_{\alpha \beta} \end{bmatrix} = \begin{bmatrix} \frac{1}{A \partial A}{\partial a} + \frac{\psi_a}{AB}{\partial A}{\partial R_a} + \frac{\psi_\beta}{AB}{\partial B}{\partial R_\beta} \\ \frac{1}{B \partial B}{\partial \beta} + \frac{\psi_\beta}{AB}{\partial B}{\partial R_\beta} + \frac{\psi_\alpha}{AB}{\partial A}{\partial R_a} \\ \frac{1}{A \partial A}{\partial \gamma} + \frac{\psi_\alpha}{AB}{\partial A}{\partial R_a} + \frac{\psi_\beta}{AB}{\partial B}{\partial R_\beta} \\ \frac{1}{B \partial B}{\partial \alpha} + \frac{\psi_\alpha}{AB}{\partial B}{\partial R_\beta} + \frac{\psi_\beta}{AB}{\partial A}{\partial R_a} \end{bmatrix}$$
\[
\begin{align*}
\{\psi^o\} = \left[ \begin{array}{l}
\gamma^o_{\alpha} \\
\gamma^o_{\beta}
\end{array} \right] = \left[ \begin{array}{l}
\gamma^o_{\alpha} = \psi^o + \frac{1}{A} \frac{\partial w}{\partial \alpha} - \frac{u^o}{R^o} \\
\gamma^o_{\beta} = \psi^o + \frac{1}{B} \frac{\partial w}{\partial \beta} - \frac{v^o}{R^o}
\end{array} \right] \quad (A6)
\end{align*}
\]

The nonlinear in-plane strains are
\[
\{\epsilon^{nl}\} = \left[ \begin{array}{c}
\epsilon_{\alpha\alpha}^{nl} \\
\epsilon_{\beta\beta}^{nl} \\
\epsilon_{\alpha\beta}^{nl}
\end{array} \right] = \left[ \begin{array}{c}
\frac{1}{2} \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} \right)^2 + \frac{1}{2} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} \right)^2 \\
\frac{1}{2} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} \right)^2 + \frac{1}{2} \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} \right)^2 \\
\frac{1}{2} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} \right) \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} \right) - \frac{1}{2} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} \right) \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} \right)
\end{array} \right] \quad (A7)
\]

\[
\begin{align*}
2 \epsilon_{\alpha\alpha}^{nl} &= \left( \frac{1}{A} \frac{\partial u^o}{\partial \alpha} + \frac{v^o}{AB} \frac{\partial A}{\partial \beta} + \frac{w}{R^o} \right)^2 \\
2 \epsilon_{\beta\beta}^{nl} &= \left( \frac{1}{B} \frac{\partial v^o}{\partial \beta} + \frac{u^o}{AB} \frac{\partial B}{\partial \alpha} + \frac{w}{R^o} \right)^2 \\
2 \epsilon_{\alpha\beta}^{nl} &= \left( \frac{1}{B} \frac{\partial v^o}{\partial \beta} + \frac{u^o}{AB} \frac{\partial B}{\partial \alpha} + \frac{w}{R^o} \right) \left( \frac{1}{A} \frac{\partial u^o}{\partial \alpha} + \frac{v^o}{AB} \frac{\partial A}{\partial \beta} + \frac{w}{R^o} \right) + \\
&\quad \left( \frac{1}{B} \frac{\partial v^o}{\partial \beta} + \frac{u^o}{AB} \frac{\partial B}{\partial \alpha} + \frac{w}{R^o} \right) \left( \frac{1}{A} \frac{\partial u^o}{\partial \alpha} + \frac{v^o}{AB} \frac{\partial A}{\partial \beta} + \frac{w}{R^o} \right) + \\
&\quad \left( \frac{1}{B} \frac{\partial v^o}{\partial \beta} + \frac{u^o}{AB} \frac{\partial B}{\partial \alpha} \right) \left( \frac{1}{A} \frac{\partial u^o}{\partial \alpha} + \frac{v^o}{AB} \frac{\partial A}{\partial \beta} \right) \frac{1}{A} \frac{\partial w}{\partial \alpha} - \frac{v^o}{R^o} \\
&\quad + \frac{1}{A} \frac{\partial w}{\partial \alpha} - \frac{v^o}{R^o} \quad (A8)
\end{align*}
\]

Nonlinear strains in a shell with initial geometrical imperfections (FSDT with von Karman’s approximation) are given by

\[
\begin{align*}
\epsilon_{\alpha\alpha}^{nl} &= \frac{1}{2} \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} \right)^2 \\
\epsilon_{\beta\beta}^{nl} &= \frac{1}{2} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} \right)^2 \\
\epsilon_{\alpha\beta}^{nl} &= \frac{1}{AB} \frac{\partial w}{\partial \alpha} \frac{\partial w}{\partial \beta}
\end{align*}
\]

Appendix B: Plate with initial imperfections.

Neglecting the transverse shear strains,

\[
\begin{align*}
\psi_x + \frac{\partial w}{\partial x} &= 0, \\
\psi_y + \frac{\partial w}{\partial y} &= 0
\end{align*}
\]

The curvatures become

\[
\{\kappa\} = \left[ -\frac{\partial^2 w}{\partial x^2}, -\frac{\partial^2 w}{\partial y^2}, -2 \frac{\partial^2 w}{\partial x \partial y} \right]^T
\]

The nonlinear strains are

\[
\begin{align*}
\epsilon_{xx}^{nl} &= \frac{1}{2} \left( \frac{1}{A} \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{1}{B} \frac{\partial w}{\partial y} \right)^2 \\
\epsilon_{yy}^{nl} &= \frac{1}{2} \left( \frac{1}{A} \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{1}{B} \frac{\partial w}{\partial y} \right)^2 \\
\epsilon_{xy}^{nl} &= \frac{1}{2} \left( \frac{1}{A} \frac{\partial w}{\partial x} \right) \frac{1}{B} \frac{\partial w}{\partial y} + \frac{1}{2} \left( \frac{1}{B} \frac{\partial w}{\partial y} \right) \frac{1}{A} \frac{\partial w}{\partial x}
\end{align*}
\]

The force and moment resultants are defined as

\[
\begin{align*}
\{N\} &= \left[ N_{\alpha\alpha}, N_{\beta\beta}, N_{\alpha\beta} \right] = \int_{-h/2}^{h/2} \left[ \sigma_{\alpha\alpha}, \sigma_{\beta\beta}, \sigma_{\alpha\beta} \right] \sigma d\zeta \\
\{M\} &= \left[ M_{\alpha\alpha}, M_{\beta\beta}, M_{\alpha\beta} \right] = \int_{-h/2}^{h/2} \left[ \sigma_{\alpha\alpha}, \sigma_{\beta\beta}, \sigma_{\alpha\beta} \right] M d\zeta
\end{align*}
\]

where \( \{\sigma\} = \left[ \sigma_{\alpha\alpha}, \sigma_{\beta\beta}, \sigma_{\alpha\beta} \right] \). For an arbitrary layer, the stress-strain relations are given by \( \{\sigma\} = [Q] \{\epsilon\} \) where \( \{\epsilon\} = \{\epsilon^o\} + \zeta \{\kappa\} + \{\epsilon^{nl}\} \) is given in Eq. 1.

With the usual A, B, D matrices,

\[
\begin{align*}
\{N\} &= [A] \{\epsilon^o\} + [B] \{\kappa\} + [A] \{\epsilon^{nl}\} \\
\{M\} &= [B] \{\epsilon^o\} + [D] \{\kappa\} + [B] \{\epsilon^{nl}\}
\end{align*}
\]

The strain energy, kinetic energy and the work done by the external forces are defined as
\[ U = \frac{1}{2} \int_{V} \{e^T \sigma \} dV + \frac{1}{2} \int_{V} \{e^T \theta \} \{r\} dV \]
\[ K = \frac{1}{2} \int_{V} \rho (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) dV \]
\[ W = \int_{\Omega} \rho w d\Omega \]

where the external pressure \( p \) and the transverse displacement \( w \) are both functions of \( \alpha, \beta, \) and \( t. \)

Using Eq. 1 and integrating through the thickness,
\[
\int_{h/2}^{h/2} \{e^T \sigma\} d\xi = \{e^T \sigma \} [A][e^{n\xi}] + \{e^{n\xi} \} [A] [e^T \sigma] + \{K\} [D][K]
\]
\[ + 2 \{e^T \sigma \} [A][e^{n\xi}] + 2 \{e^{n\xi} \} [B][K] + 2 \{K\} [B][e^{n\xi}] \]

For symmetric laminates \([B]=0\) and the last two terms disappear. With the approximations made here (Eqs.), \( \{e^{n\xi}\}=0 \) and several other terms drop out and the strain energy becomes
\[
U = \frac{1}{2} \int_{\omega} \left( \{e^{n\xi}\} [A][e^{n\xi}] + \{K\} [D][K] \right) d\Omega \tag{B4}
\]

**Appendix C: Cylindrical shells**

\[
D_1 \frac{\partial^4 w}{\partial \xi^4} + 2(D_{12} + 2D_{66}) \frac{1}{R^2} \frac{\partial^2 w}{\partial \xi^2 \partial \theta^2} +
\]
\[
D_{22} \frac{1}{R^2} \frac{\partial^4 w}{\partial \theta^2} + \frac{1}{R} \frac{\partial^2 \phi}{\partial \xi^2} + \tilde{p} \dot{w} = q \tag{C1}
\]

\[
\frac{A_1^2 - A_1 A_{22}}{R \partial^2} + \frac{A_{22}^4}{\partial^2} + \frac{A_{22}}{R^4} \frac{\partial^4 \phi}{\partial \theta^4} +
\]
\[
\frac{A_1 A_{22} - A_{22}^2}{\partial^2} - 2A_{22} A_{66} \frac{\partial^4 \phi}{\partial \xi^2 \partial \theta^2} = 0 \tag{C2}
\]

Eq. 1 is identical to the equation of motion for an orthotropic plate except for the coupling between inplane and transverse displacements introduced by the \( \frac{1}{R} \frac{\partial^2 \phi}{\partial \xi^2} \) term. Natural frequencies are given by
\[
\alpha_{22} \left( \alpha_{11} - \frac{\alpha_{12}^2 \xi_{21}}{\alpha_{22}} \right) \left( \rho h \right) \tag{C3}
\]
where, for a simply supported cylindrical panel,


Fig. 3: Phase portrait for nonlinear system subjected to a step load with $\varepsilon = 16$ (Solid line: numerical, dotted line: approximate, dashed line: impulsive response).

Fig. 4: Response to a rectangular pulse for several values of $\varepsilon$. Blue line: impulsive response to a step load; red line impulsive response to a step load.

Fig. 5: Non-dimensional restoring force $F_R$ as a function of $x$ for three values of $\varepsilon$ and $x_0=1$. Solid line: $\varepsilon = 1$, dashed line: $\varepsilon = 2$, dotted line: $\varepsilon = 4$.

Fig. 6: Phase portrait for the initial value problem of a system governed by Eq. (12) with $f(\tau) = 0$ and $x_0 = 1$. Solid line: $\varepsilon = 0$, dashed line: $\varepsilon = 1$, dotted line: $\varepsilon = 2$. 
Fig. 7: Non-dimensional displacement $x$ versus time $\tau$ for a linear single degree of freedom system subjected to an exponential pulse $\tau_o = 10$

Fig. 8: Phase portrait ($\dot{x}$ versus $x$) for a linear single degree of freedom system subjected to an exponential pulse $\tau_o = 10$

Fig. 9: Phase portrait ($\dot{x}$ versus $x$) for a linear single degree of freedom system subjected to an exponential pulse $\tau_o = 0.5$

Fig. 10: Effect of the mass ratio $m_p/m$ on the nondimensional force history
**Table 1:** Maximum displacements and velocities as a function of $\varepsilon$ for Duffing’s equation with step load.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$x_{max}$</th>
<th>$x_0$</th>
<th>$x'_{max}$</th>
</tr>
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<tr>
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**Table 2:** Comparison of natural frequencies as a function of $\varepsilon$ obtained numerically and using Eq. 11.

<table>
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<th>$\varepsilon$</th>
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<th>Approximate</th>
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**Table 3:** Natural frequencies of stainless steel cylindrical shells (R= 1 m, L=20 m, h=2 mm) for $m=1$.

- $\omega_{app}^S$: approximate Soedel solution; $\omega^S$: Soedel’s solution; $\omega^L$: results from Loy et al [24]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\omega_{app}^S$</th>
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<th>$\omega^L$</th>
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