SMP FILLED HONEYCOMB AS A RECONFIGURABLE SKIN: MODEL AND EXPERIMENTAL VALIDATION

R.V. Beblo1*, J.P. Puttmann1, N.E. DeLeon2, J.J. Joo3, G.W. Reich3
1 University of Dayton Research Institute, Dayton, OH, USA
2 AFRL/RQVV, Wright-Patterson Air Force Base, USA
3 AFRL/RQVC, Wright-Patterson Air Force Base, USA
* Corresponding author (Richard.Beblo@udri.udayton.edu)

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1 General Introduction
Reconfigurable structures have been a topic of much research in recent years including candidate materials and systems used to cover them, aka ‘skins’. It has also been proposed that the skin covering structures such as the wing of a reconfigurable aircraft, satellite, wind turbine, or ship hull could be made of a cellular structure filled with a variable stiffness material, such as honeycomb filled with shape memory polymer (SMP). With individually addressable and defined cells a skin of honeycomb filled with SMP could be designed to provide a wide range of structural properties. [1-5]

To aid in the design and optimization of such a cellular skin, it is helpful to develop an accurate analytical model of the system. Such a model could be used to calculate in-plane composite mechanical properties such as Young’s modulus, shear modulus, and Poisson’s ratio which can in turn be used to calculate deflection of the skin under load. While there are many models that predict the properties of unfilled honeycomb, see Table 2, few predict the properties of filled honeycomb. Moreover, those that do are only accurate at very low infill moduli.

For comparative purposes, four models predicting the unfilled Young’s modulus of honeycomb following the general Eqs. (1) are listed in Table 2 with variable definitions shown in Fig. 1 and Table 1.

\[
E_{c\theta} = B_1 E_h + B_2 E_i \\
E_{c\pi/2} = D_1 E_h + D_2 E_i
\]

(Eq. 1)

\(E_{c\theta}\) and \(E_{c\pi/2}\) are Young’s moduli in the \(\theta\) and \(\pi/2\) directions of the composite (or honeycomb if \(E_i\) is zero), \(E_h\) and \(E_i\) are the Young’s moduli of the honeycomb and infill material, and \(B_1, B_2, D_1,\) and \(D_2\) are geometric coefficients listed in Table 2.

Table 1. Experimental honeycomb geometry

<table>
<thead>
<tr>
<th>Geometry Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) (mm)</td>
<td>5.251</td>
</tr>
<tr>
<td>(l) (mm)</td>
<td>9.334</td>
</tr>
<tr>
<td>(d) (mm)</td>
<td>0.044</td>
</tr>
<tr>
<td>(c) (mm)</td>
<td>6.387</td>
</tr>
<tr>
<td>(\theta) (radians)</td>
<td>0.7987</td>
</tr>
</tbody>
</table>

Table 2 lists several models predicting Young’s modulus for unfilled honeycomb. The first set of coefficients, by Masters and Evans, calculates the modulus of empty honeycomb by first calculating deflection, axial strain, and shear using beam theory.
separately for each member within the unit cell before combining them into a single equation. [6] This results in the lowest predicted Young’s modulus for the studied honeycomb geometry listed in Table 1; see Table 3. The remaining models for unfilled honeycomb are variations of this theory, differing only in the assumptions used and neglected terms.

Table 2. Unfilled honeycomb model coefficients

<table>
<thead>
<tr>
<th></th>
<th>B₁</th>
<th>D₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>Masters and Evans*</td>
<td>(y_0 \left( \frac{y_0}{d^3} + \frac{E_y y_0^2}{G_y l} + \frac{2a l + x_0^2}{d l} \right))</td>
<td>(y_0 \left( \frac{x_0^2 l}{d^3} + \frac{E_y x_0^2}{G_y l} + \frac{y_0^2}{d l} \right))</td>
</tr>
<tr>
<td>Abd El-Sayed</td>
<td>(y_0 \left( \frac{y_0}{d^3} + \frac{a}{l d} + \frac{a}{d} \right))</td>
<td>(y_0 \left( \frac{x_0^2 l}{d^3} \right))</td>
</tr>
<tr>
<td>Gibson and Ashby simplified</td>
<td>(y_0 \left( \frac{y_0}{d^3} \right))</td>
<td>(y_0 \left( \frac{x_0^2 l}{d^3} \right))</td>
</tr>
<tr>
<td>Gibson and Ashby full**</td>
<td>(y_0 \left( \frac{y_0}{d^3} + \frac{x_0^2}{2 l d} + \frac{a}{d} + \frac{\kappa y_0^2}{2 l d} \right))</td>
<td>(y_0 \left( \frac{x_0^2 l}{d^3} + \frac{\kappa y_0^2}{2 l d} + \frac{y_0^2}{2 l d} \right))</td>
</tr>
</tbody>
</table>

* \(G_y = \) Honeycomb Shear Modulus
** \(\kappa = 2.4 + 1.5\nu_y\)

Table 3. Unfilled honeycomb moduli

<table>
<thead>
<tr>
<th></th>
<th>Young’s Modulus (Pa)</th>
<th>(E_0)</th>
<th>(E_{\infty})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Masters and Evans</td>
<td>2.62e4 8.94e3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Abd El-Sayed</td>
<td>2.62e4 8.95e3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gibson and Ashby (simple)</td>
<td>2.62e4 8.95e3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gibson and Ashby (full)</td>
<td>7.15e4 2.10e5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For filled honeycomb, Abd El-Sayed et al derived a mechanics model using internal strain energy of the infill and Castiglione’s methods for deformation of the honeycomb, the corresponding coefficients for which are listed in Table 4. [8] A composite modulus is calculated by forcing the global deformation of the honeycomb and infill to be equal. Their work is the earliest presented and the basis for many other honeycomb models, including the often cited models by Gibson and Ashby. The work predicts Young’s modulus in two directions as well as Poisson’s ratio in the elastic, elasto-plastic, and plastic deformation regions for both unfilled and filled honeycomb. The model is shown to correlate well with experimental data with an infill modulus more than four orders of magnitude lower than that of the honeycomb. The equations shown in Table 2 and Table 4 were re-derived using the methods of Abd El-Sayed due to typos in the original articles.

Table 4. Filled honeycomb model coefficients

<table>
<thead>
<tr>
<th></th>
<th>B₂</th>
<th>D₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abd El-Sayed</td>
<td>(K_y x_0^2 (a + x_0) ) (2y_0^3(1 - \nu_y^2))</td>
<td>(K_y y_0) (2(a + x_0)(1 - \nu_y^2))</td>
</tr>
</tbody>
</table>

\[K = \frac{y_0^3}{x_0^3} \left( \frac{a}{x_0^3} + \frac{\alpha y_0^3}{2x_0^3} \ln \left( 1 + \frac{2x_0}{a} \right) \right) \frac{x_0}{y_0} + \frac{\alpha}{y_0} + \frac{2\nu y_0}{x_0}\]

There have been numerous other works presented involving the modeling and experimental and FEA validation of filled and unfilled honeycomb and other cellular structures, some of which include those by: Burton [9], Zhu [10], Murray [11], Najamoto [12], Klintworth [13], Burlayenko [14], Schwinghacker [15], Henry [16-17], Guo [18], and Chen [19-20]. Due to the complexity of modeling filled honeycomb analytically however, these studies have been limited to FEA analysis which have limited use in optimization.
2 Model Derivation

The representative skin proposed here is comprised of aluminum alloy 3003 honeycomb purchased from McMaster-Carr filled with an epoxy SMP developed by General Motors. [21] The epoxy SMP has a hard state Young’s modulus, 1.04E9 Pa, approximately one and a half orders of magnitude lower than that of aluminum, 7.31E10 Pa, and a soft state modulus, 1.65E7, three and a half orders of magnitude lower. The geometric parameters of the aluminum honeycomb are listed in Table 1.

Filled honeycomb models available in the literature calculate the internal strain energy of the honeycomb and infill materials separately. If the global strain of each component is forced to be equal, the total strain energy can be calculated which can then be used to calculate constitutive properties of the composite. However, this approach neglects the mechanistic nature of the composite, specifically the mechanical advantage of the force exerted on the slender beams of the honeycomb by the infill under strain. To account for these mechanistic effects, the proposed model applies forces generated by strain in the infill directly to the beams of the honeycomb. Castigliano’s method is then used to predict the force-displacement relationship of the beams within the unit cell. Thus, the internal strain energy of the infill is not calculated directly; rather the effect of the infill on the deformation of the honeycomb is calculated. The tensile modulus of the composite can then be calculated knowing the dimensions of the unit cell.

In Equations (2) through (5), the subscript \( c \) represents the composite while the subscripts 0, \( \gamma/2 \), \( \theta \), and \( \theta+\gamma/2 \) designate the angle of interest \( \alpha \), shown in Fig. 1. The full derivation of Eqns. (2) through (5) including the force-displacement relations is shown in Appendix 1.

It should be noted, however, that the selection of a unit cell with periodic boundary conditions is neither arbitrary nor its existence guaranteed for all angles \( \alpha \) or all geometries. The proposed method is subject to the geometric limitations listed in Eqns. (6).

\[
\begin{align*}
\frac{a}{l} & \leq \left[ \frac{\sin(\theta)}{\sin(\theta)\cos(\theta)} \right. \\
& \quad + \frac{2\sin(\theta)\cos(\theta)}{\sin(\theta)\cos(\theta)} \\
& \quad \left. \frac{0}{\cos(\theta)} \right]
\end{align*}
\]

Eqns. (6) are generated by restricting the unit cell to encompassing one full honeycomb including one member of length \( a \), width \( d_2 \), and two members of length \( l \), width \( d_1 \). At extreme angles of \( \theta \) and ratios of all this criteria cannot be satisfied. An example of a non-compliant geometry is shown in Fig. 2 with the non-compliant part highlighted.

Fig. 2. Non-compliant honeycomb geometry
3 Experimental Procedures
To validate the derived model, the Young’s modulus of the composite was obtained experimentally for several directions. All samples were nominally 130 mm wide, 250 mm long, and 6.4 mm thick. The empty aluminum honeycomb was cut to size using scissors and a template and etched for 10 minutes in a 17.5% Nitric acid solution bath to promote adhesion with the polymer. A 70% Nitric acid-water solution was diluted 3-1 with water. The etched aluminum was then placed in a Teflon® mold and a small amount of polymer, mixed per GE’s instructions [21], poured into the bottom to create a seal isolating the individual cells of the honeycomb. The composite was then placed in a 100 °C oven for 1 hour. Additional SMP was then poured into alternating rows of cells filling the honeycomb and placed back in the oven for an additional hour. Finally, the remaining cells were filled with SMP and the complete composite was cured at 100 °C for 3 more hours. This multi-step filling process differs from GM’s proposed cure procedure and was developed to prevent cracking or delamination of the SMP due to contraction during curing. The width, thickness, and gage length of each sample was measured with calipers or a micrometer. The width of empty honeycomb samples was estimated as the average of the widths of each row of complete cells. Partial cells at the edges of the empty honeycomb were not included in the width of the sample.

Tensile tests were conducted at 4 mm/min, resulting in a strain rate of approximately 1.8 %/min, on a screw-driven MTS 1K load frame with a 250 N MTS load cell. Above $T_g$ composite tests were conducted with an electrical resistance heater, house compressed air, and a custom thermal chamber. K-type thermocouples in the thermal chamber and attached to the sample were used to monitor temperature. Tests were conducted when the sample temperature became stable at 100 °C, approximately 35 °C above $T_g$.

4 Model and Experimental Results
Interpolating between Eqns. (2) through (5) yields Young’s modulus of the composite in any direction. Fig. 3 and Fig. 4 show the predicted results for the empty honeycomb and filled composite above $T_g$ respectively.

In Fig. 3 the empty blue circles are the moduli predicted by Eqns. (2) through (5), the blue line is a cubic interpolation between Eqns. (2) through (5), and the solid red dots are experimental data with their associated 95% confidence intervals. Due to symmetry, each of the four data points from Eqns. (2) through (5) are reflected about the 0 and $\pi/2$ axis. Since $\theta$ for the experimental aluminum honeycomb is nearly $\pi/4$, the two data points representing $\alpha = \theta$ and $\alpha = \theta + \pi/2$ when reflected due to symmetry are almost identical. While the same principle of symmetry could also be used to reflect the experimental data about the 0 and $\pi/2$ axis, for simplicity they are not. Much of the large errors in the experimental data are believed to be due to the nature of the honeycomb having irregular sides and variability of the honeycomb member thickness, $d$. The modulus of the honeycomb is highly sensitive to the thickness of the members which was found to vary between honeycomb sheets as well as within each sample by as much as 25%. The average thickness, $d$, of all samples tested at $\alpha = \pi/6$ is higher than the average of all samples at $\alpha = \pi/4$ by 3% which could contribute to the discrepancy between the experimental data and presented model. An angle of maximum modulus of $\alpha = \theta$, as predicted by the model, is expected. Aside from typical errors associated with the precision of the measurement, residual stresses within the polymer as a result of shrinkage during curing could also be a significant source of variation.
The range of moduli predicted for the honeycomb/SMP composite above T_g, Fig. 4, is much less than that of the empty honeycomb. The SMP filling in effect even out the directional dependence of the stiffness of the honeycomb, making it closer to isotropic. Experimentally determined moduli in the 0 and π/2 directions correlate well with the model.

With the infill SMP below T_g, the predicted modulus distribution is identical to that shown in Fig. 4 only with the magnitude shifted higher indicating a stiffer material, Fig. 5.

Experimentally, the modulus of the composite below T_g was found to be nearly identical in the 0 and π/2 directions. This is expected, since as the modulus of the infill approaches the modulus of the honeycomb, the result is a solid sheet of uniform stiffness which would result in a perfect circle when plotted as in Fig. 5. This indicates that as the infill modulus increases, the assumption that the major mode of deformation is bending of the honeycomb members becomes invalid and a different deformation mechanism and different set of governing equations dominates the response of the composite. It can be inferred then that the infill modulus of the composite in the proposed model must be more than one and a half orders of magnitude lower than the honeycomb modulus. It is also shown that the equations are accurate for infill moduli at least three and a half orders of magnitude lower than the honeycomb modulus.

Table 5 and Table 6 list the predicted moduli for the empty and filled honeycomb in both the hard and soft states as predicted by Abd El-Sayed, the authors, and found experimentally.

### Table 5. Empty honeycomb moduli

<table>
<thead>
<tr>
<th>α</th>
<th>0</th>
<th>π/12</th>
<th>π/6</th>
<th>π/4</th>
<th>π/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abd El-Sayed</td>
<td>2.62e4</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
<td>8.95e3</td>
</tr>
<tr>
<td>Beblo</td>
<td>2.62e4</td>
<td>2.19e5</td>
<td>5.43e5</td>
<td>6.63e5</td>
<td>8.95e3</td>
</tr>
<tr>
<td>Experimental</td>
<td>6.28e4</td>
<td>5.4±1.9e5</td>
<td>7.3±1.2e5</td>
<td>6.4±0.5e5</td>
<td>1.66e4</td>
</tr>
</tbody>
</table>
Table 6. Filled honeycomb composite moduli

<table>
<thead>
<tr>
<th>Temp °C</th>
<th>Young’s Modulus (Pa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Abd El-Sayed</td>
<td>100 5.29E7</td>
</tr>
<tr>
<td></td>
<td>20  3.33E9</td>
</tr>
<tr>
<td>Beblo</td>
<td>100 3.13E7</td>
</tr>
<tr>
<td></td>
<td>20  1.97E9</td>
</tr>
<tr>
<td>Experimental</td>
<td>100 3.0E7</td>
</tr>
<tr>
<td></td>
<td>20  2.22E9</td>
</tr>
</tbody>
</table>

It should be noted that all of the experimental and model predicted Young’s moduli listed thus far have been for an irregular honeycomb shape, see Table 1. For a regular honeycomb, where $a = l$, $\theta = \pi/6$, and $d \ll a, l, c$ all of the models listed in Table 2 as well as the presented model predict a nearly isotropic material for both an empty honeycomb and composite. If the thickness of the honeycomb members, $d$, approaches $a$, $l$, or $c$ then the material is not nearly isotropic.

5 Conclusions

Although there remain sources of error, the presented model predicts the tensile modulus of filled honeycomb better than models currently available in the literature and can be used to predict the modulus at any angle. The model is limited as the infill modulus approaches that of the honeycomb; however this occurs typical volume averaging techniques can be employed.

Although there is significant uncertainty in the experimental data, particularly for the unfilled honeycomb, much of the associated error is unavoidable, deriving from variability in manufacturing and the nature of the geometry of the material. Since the model is analytic and differentiable with respect to the honeycomb geometric parameters $a$, $l$, $d$, and $\theta$; the model has the potential to serve as an excellent optimization tool. The model can be used to predict the modulus of a filled honeycomb composite in any direction and for any honeycomb geometry, given certain constraints such as thin beams and an infill modulus in the acceptable range. As such, the model can be used in design to solve for the optimal honeycomb geometry given desired deformation and stiffness requirements.

6 Acknowledgements

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References


Appendix I: Young’s Modulus Derivation
Assuming the materials are linearly elastic, Castigliano’s method is used to calculate deflection of the composite; which is then used to calculate the tensile modulus. Castigliano states that the force-deflection relationship of the honeycomb is:

\[
\delta_j = \sum_m \left\{ \int_0^L \frac{N_m^2}{2E_mA_m} \frac{\partial^2}{\partial z^2} + \int_0^L \frac{M_{z,m}^2}{2E_m I_{z,m}} \frac{\partial}{\partial z} \right\} 
\]

where \( \delta \) is the deflection in the direction and at the point of the applied force \( F_j \), \( m \) is the number of members, \( N \) is the axial load, \( E \) is Young’s modulus, \( A \) is the cross-sectional area of the member, \( M \) is the moment, \( I \) is the moment of inertia, \( k \) is a geometric correction coefficient, \( V \) is the shear force, \( G \) is the shear modulus, \( T \) is the torque, and \( J \) is the polar moment of inertia. The subscript \( j \) represents the direction of the deflection being sought; in this case either \( 0, 7\alpha, \) or \( \theta \) shown as \( \alpha \) in Fig. 1.

With the unit cell corresponding to an \( \alpha \) of 0 shown in Fig. 1, the forces on the slanted and axial honeycomb segments within the unit cells are similar to those shown in Fig. 6 and Fig. 7.

Where the subscript \( i \) indicates a force generated by straining the infill and the subscript \( \theta \) refers to an applied force. The moment \( m_0 \) is found such that a fixed end condition is imposed. The forces due to the infill in each of the four directions are:

\[
F_{0x} = m_0 
\]

\[
F_{0\alpha} = F_{\alpha i} 
\]

\[
F_{\alpha r} = F_{\alpha y} 
\]
\[ F_{\alpha}^{0}(s) = \frac{\delta^0E_i(l - 2s)}{l(a + 2x_0)} = F_{\alpha}^{0'} \delta^0(l - 2s) \]

\[ F_{\alpha x}^{0} = \frac{\delta^0E_i x_0 a}{y_0^2} = F_{\alpha x}^{0'} \delta^0 \quad (8) \]

\[ F_{\beta l}^{0}(s) = \frac{\delta^0E_i x_0^2}{by_0^2} = F_{\beta l}^{0'} \delta^0 \]

\[ F_{\alpha}^{\theta}(s) = \frac{\delta^\theta E_i(a - 2s)}{2ay_0} = F_{\alpha}^{\theta'} \delta^\theta(a - 2s) \]

\[ F_{\alpha l}^{\theta}(s) = \frac{\delta^\theta E_i(l - 2s)}{l(a + 2x_0)} = F_{\alpha l}^{\theta'} \delta^\theta(l - 2s) \]

\[ F_{\alpha y}^{\theta}(s) = \frac{\delta^\theta E_i y_i(x_i y_2 + x_2 y_i)}{4ay_0 y_1 y_2} = F_{\alpha y}^{\theta'} \delta^\theta \quad (9) \]

\[ F_{\beta l}^{\theta}(s) = \frac{\delta^\theta E_i x_2(x_i y_2 + x_2 y_i)}{4ly_0 y_1 y_2} \]

\[ F_{\theta l}^{\theta}(s) = \frac{\delta^\theta E_i l(x_i y_2 + x_2 y_i)}{2y_0 y_1 y_2} = F_{\theta l}^{\theta'} \delta^\theta \]

\[ F_{\alpha}^{\theta+\tau/2}(s) = \frac{\delta^{\theta+\tau/2}E_i y_i y_2(a - 2s)}{2ay_0 (x_i y_2 + x_2 y_i)} \]

\[ = F_{\alpha}^{\theta+\tau/2} \delta^{\theta+\tau/2}(a - 2s) \]

\[ F_{\alpha l}^{\theta+\tau/2}(s) = \frac{\delta^{\theta+\tau/2}E_i y_i y_2(l - 2s)}{l(a + 2x_0)(x_i y_2 + x_2 y_i)} \]

\[ = F_{\alpha l}^{\theta+\tau/2} \delta^{\theta+\tau/2}(l - 2s) \]

\[ F_{\alpha y}^{\theta+\tau/2}(s) = \frac{\delta^{\theta+\tau/2}E_i x_i}{4ay_0} = F_{\alpha y}^{\theta+\tau/2} \delta^{\theta+\tau/2} \quad (10) \]

\[ F_{\alpha l}^{\theta+\tau/2}(s) = \frac{\delta^{\theta+\tau/2}E_i x_2}{4ay_0} = F_{\alpha l}^{\theta+\tau/2} \delta^{\theta+\tau/2} \]

\[ F_{\beta l}^{\theta+\tau/2}(s) = \frac{\delta^{\theta+\tau/2}E_i l}{2y_0} = F_{\beta l}^{\theta+\tau/2} \delta^{\theta+\tau/2} \]

\[ F_{\alpha}^{\theta+\tau/2}(s) = \frac{\delta^{\theta+\tau/2}E_i y_0 (l - 2s)}{x_0 l(a + 2x_0)} = F_{\alpha}^{\theta+\tau/2} \delta^{\theta+\tau/2}(l - 2s) \]

\[ F_{\alpha y}^{\theta+\tau/2} = \frac{\delta^{\theta+\tau/2}E_i a}{2y_0} = F_{\alpha y}^{\theta+\tau/2} \delta^{\theta+\tau/2} \]

\[ F_{\alpha l}^{\theta+\tau/2} = \frac{\delta^{\theta+\tau/2}E_i x_0}{4ly_0} = F_{\alpha l}^{\theta+\tau/2} \delta^{\theta+\tau/2} \]

The subscript \( i \) indicates a force due to the infill, \( x \) or \( y \) indicates the direction of the force, \( a \) or \( l \) indicates the type of member the force acts on, either slanted or horizontal, and the final subscript indicates the orientation of the unit cell, \( \alpha \). The slanted member \( l1 \) in the \( \theta \) and \( \theta + \pi/2 \) directions is the member parallel to the \( \alpha \) direction while the member designated \( l2 \) is the remaining slanted member. The remaining dimensions are:

\[ x_0 = l \cos(\theta) \]
\[ y_0 = l \sin(\theta) \]
\[ x_i = a \cos(\theta) \]
\[ y_i = a \sin(\theta) \]
\[ x_2 = l \cos(\pi - 2\theta) \]
\[ y_2 = l \sin(\pi - 2\theta) \]

\[ \delta^{\theta+\tau/2} = \delta^\theta \frac{2x_0}{y_0} \]

\[ \delta^{\theta+\tau/2} = \delta^\theta \frac{(x_i y_2 + x_2 y_i)}{y_1 y_2} \]

\[ n = \frac{\sin(\pi - 2\theta) \cos(\theta)}{\sin(\theta) \cos(\pi - 2\theta) + \sin(\pi - 2\theta) \cos(\theta)} \]

Decomposing these forces into directions axial and perpendicular to the slanted members result the components shown in Fig. 8.
The subscripts \( a, l, l_1, \) and \( l_2 \) indicate the respective honeycomb members within the unit cell while the subscripts \( r \) and \( s \) indicate direction as shown in Fig. 7 and Fig. 8. In Eqns. (13) and (14) symmetry yields \( \delta_{ij} \) and \( \delta_{il} \) are equivalent, thus only one needs to be solved for and a factor of 2 applied. The lower case forces in Eqns. (13) through (16) correspond to the decomposed forces in Eqns. (8) through (11) in the \( r \) and \( s \) coordinate system. The individual deflection contributions of each member are found using Castigliano’s method, Eqn. (7). The resulting total deflections of the unit cell in the four desired directions are then:

\[
\delta^0 = \delta_{11r}^0 (f_{0r}^0) + \delta_{12r}^0 (f_{02r}^0, f_{y01r}^0, f_{y02r}^0, f_{z02r}^0) + \delta_{12s}^0 (f_{02s}^0, f_{y01s}^0, f_{y02s}^0, f_{z02s}^0) + \delta_{1s}^0 (f_{0s}^0, f_{y01s}^0, f_{y02s}^0, f_{z02s}^0) (15)
\]

\[
\delta^0 = \delta_{11s}^0 (f_{0s}^0) + \delta_{12s}^0 (f_{02s}^0, f_{y01s}^0, f_{y02s}^0, f_{z02s}^0) + \delta_{12r}^0 (f_{02r}^0, f_{y01r}^0, f_{y02r}^0, f_{z02r}^0) + \delta_{1r}^0 (f_{0r}^0, f_{y01r}^0, f_{y02r}^0, f_{z02r}^0) (16)
\]
\[ \delta^\alpha = F_0^\alpha \left[ \frac{1}{Ecd_1} + \frac{n^ix_1}{Ecd_2} + \frac{n^iy_1^2}{Ecd_4} + \frac{(1-n)^2}{Ecd_1} + \frac{(1-n)^2x_1^2}{Ecd_2} \right] - F_{\theta l}^\alpha , \delta^\alpha \left[ \frac{max_{1,2}}{Ecd_1} + \frac{(1-n)x_1y_1}{Ecd_4} - \frac{nx_1y_1}{Ecd_4} - (1-n)x_1y_2 \right] \]

\[ -F_{\theta l}^\alpha , \delta^\alpha \left[ \frac{na^2x_1y_1}{2Ecd_2} - \frac{nx_1y_1}{2Ecd_2} \right] - F_{\theta l}^\alpha , \delta^\alpha \left[ \frac{(1-n)x_1y_1^2}{6Ecd_1} + \frac{(1-n)y_1^2}{5Ecd_1} \right] \]

\[ -F_{\theta l}^\alpha , \delta^\alpha \left[ \frac{na^2}{6Ecd_2} + \frac{na^2y_1}{5Ecd_2} - \frac{na^2x_1y_1}{6Ecd_2} \right] - F_{\theta l}^\alpha , \delta^\alpha \left[ \frac{na^2y_1}{2Ecd_1} \right] \]

\[ \delta^{\theta + \gamma_2} = F_0^{\theta + \gamma_2} \left[ \frac{y_1^2}{Ecd_2a} + \frac{a^2y_1^2}{Ecd_3} + \frac{y_2^2}{Ecd_3} + \frac{y_1y_2}{Ecd_3} \right] - F_{\theta l}^{\theta + \gamma_2} , \delta^{\theta + \gamma_2} \left[ \frac{ax_1^2}{Ecd_2} + \frac{a^2x_1^2}{Ecd_3} + \frac{y_1^2}{Ecd_3} + \frac{y_2^2}{Ecd_3} \right] \]

\[ -F_{\theta l}^{\theta + \gamma_2} , \delta^{\theta + \gamma_2} \left[ \frac{y_1^2}{2Ecd_2} + \frac{a^2y_1^2}{2Ecd_3} \right] - F_{\theta l}^{\theta + \gamma_2} , \delta^{\theta + \gamma_2} \left[ \frac{y_1^2}{2Ecd_1} + \frac{a^2y_1^2}{2Ecd_3} \right] \]

\[ -F_{\theta l}^{\theta + \gamma_2} , \delta^{\theta + \gamma_2} \left[ \frac{ax_1y_1}{5Ecd_2} - \frac{ax_1y_1}{6Ecd_2} \right] + F_{\theta l}^{\theta + \gamma_2} , \delta^{\theta + \gamma_2} \left[ \frac{l^3x_1y_1^2}{5Ecd_2} - \frac{lx_1y_2}{6Ecd_2} \right] \]

Notice that Eqns. (17) through (20) are not solved for the deflections explicitly for clarity since the forces due to the infill are dependent upon deflection. The tensile modulus in each of the four desired directions is then (repeated from previous)

\[ E_{\theta 0} = \frac{F_{00}}{\delta_0} \left( \frac{l + x_1}{y_0} \right) \]

\[ E_{\theta \gamma_2} = \frac{F_{\theta l}}{\delta_{\gamma_2}} \left( 2y_0 \frac{c(a + x_0)}{c(l + x_1)} \right) \]

If the effect of the infill is ignored, Eqns. (2) and (4) can be simplified and shown to be equivalent to those of Abd El-Sayed shown in Table 2. A spline interpolation between Eqns. (2) through (5) is then used to estimate the modulus of the honeycomb/SMP composite in any direction, resulting in Fig. 3.

Similar methods can also be used to predict the shear modulus of the composite as well as Poisson’s Ratio.