1 Introduction

Hyperelastic and hypoelastic constitutive law are proposed to describe the mechanical behaviour of fibre bundles of woven composite reinforcements. The objective of these models is to compute the 3D geometry of the deformed woven unit cell of a composite reinforcement during a forming process. This geometry is important for permeability calculations [1] and for the mechanical behaviour of the composite into service. The finite element models of a woven unit cell can also be used as virtual mechanical tests. The highlight of four deformation modes of the fibre bundle leads to definition of a strain energy potential from four specific invariants used in a hyperelastic approach [2]. The parameters of the hyperelastic constitutive law are identified thanks to uniaxial and equibiaxial tensile tests on the fibre bundle and on the whole reinforcement. This constitutive law is then validated in comparison to biaxial tension and in-plane shear tests.

On the other hand an hypoelastic model will also be presented to depict the mechanical behaviour of the yarn. It is based on the fibre rotation. The compression responses of several layer stacks with parallel or different orientations will be analysed [3]. The numerical simulations show good agreement when compared to compaction experiments. These mesoscopic simulations can be used as virtual compression tests. In addition they determine the internal geometry of the reinforcement after compaction. There too, this internal geometry is important to compute the permeability of the deformed reinforcement and to calculate the homogenised mechanical properties of the final composite part [4].

Some arguments on the respective advantages and drawbacks of hypoelastic and hyperelastic models will be discussed.

2. Hyperelastic model for a fibre bundle

2.1. Transverse isotropy

Fig. 1. Transverse isotropy of the yarn in a textile reinforcement. X-ray tomography imaging.

The yarn is considered as a continuum. The homogenized material has a preferred direction which is the fibre one. The spatial distribution of fibres inside the yarn has been analysed on deformed cross sections [5]. It has been concluded that this distribution is isotropic in the plane. The “yarn” material will then be assumed to be transversely isotropic.

2.2. Deformation modes

Fig. 2. Deformation modes of the yarn:
(a) Elongation (b) Compaction of the cross section (c) Distortion of the section (d) Longitudinal shear.
Four deformation modes are considered: the
elongation in the fibre direction, the compaction in
the transverse section of the yarn, the distortion
(shear) in the transverse section and the shear along
the fibre direction (Fig. 2.). The last one
(longitudinal shear) is a measure of the angle
between the covariant cross section and the initial
cross section. It is then compound of both
longitudinal shear deformations. This deformation
mode mainly controls the bending rigidity of the
yarn.

2.3 Finite displacement kinematics
Let \( X \) be the initial position of a material point and
\( x \) its position in the current configuration. \( \chi \) is the
transformation from the initial to the current
configuration: \( x = \chi(X) \). The deformation
gradient tensor is then:

\[
F = \frac{\partial x}{\partial X}
\]  

(1)

This tensor allows the transformation of any vector
\( A \) in the initial configuration into a vector \( a \) of the
current (deformed) configuration. The Cauchy-
Green deformation tensor \( C = F^T F \), from which
the variation of the length of \( A \) can be deduced, is
defined as follows:

\[
\|a\|^2 = (F \cdot A) \cdot (F \cdot A) = A \cdot (F^T \cdot F) \cdot A = A \cdot C \cdot A
\]  

(2)

The invariants of tensor \( C \) are usually defined by:

\( I_1 = \text{trace}(C) \), \( I_2 = \frac{1}{2} \left( \text{trace}(C)^2 - \text{trace}(C^2) \right) \),
\( I_3 = \det(C) \)  

(3)

The local change of volume during deformation is
described by the Jacobian \( J = \sqrt{I_3} = \det(F) \).

2.4. Hyperelastic equations

In order to define a hyperelastic material, an elastic
strain energy potential per unit volume \( w \) must exist
which only depends on the current strain state. As
the behaviour is reversible, it does not dissipate
energy and naturally fulfils the Clausius-Duhem
inequality. These assumptions allow the constitutive
equation of a hyperelastic material to be written as:

\[
S(F) = 2 \frac{\partial w(F)}{\partial C}
\]  

(4)

where \( F \) is the deformation gradient tensor and \( S \)
is the second Piola-Kirchhoff stress tensor.

To ensure that this constitutive equation fulfils the
principle of material frame indifference, a function
\( \tilde{w} \) of the right Cauchy Green strain tensor \( C \) must
exist as [6]:

\[
\tilde{w}(F^T F) = \tilde{w}(C) = w(F)
\]  

(5)

Consequently, the strain energy potential can be
defined directly, using \( \tilde{w}(C) \). Moreover, it must
satisfy the principle of material symmetry, i.e. the
strain energy must be invariant under any
transformation belonging to the symmetry group \( G \)
of the material:

\[
\tilde{w}(C) = \tilde{w}(Q C Q^T) \quad \forall Q \in G
\]  

(6)

The symmetry group of an isotropic material
includes all the rotations and reflections (i.e.
\( G = O_3 \), the group of isometries of \( \mathbb{R}^3 \).
Consequently, the strain energy \( \tilde{w} \) is
mathematically isotropic. For an anisotropic
material, \( G \subset O_3 \) so \( \tilde{w} \) is mathematically
anisotropic. In such a case, it has been shown [7, 8]
that an isotropic function \( \hat{w} \) exists which depends
on the right Cauchy-Green strain tensor \( C \) and on
\( N \) structural tensors \( G_i \) defining the symmetry
group, such as:

\[
\hat{w}(C) = \hat{w}(C, G_1, ..., G_N)
\]  

(7)

The representation theorems for transversely
isotropic functions of scalars, vectors and/or
symmetric tensor variables define a finite number of
scalar functions \((I_1, ..., I_n)\) and a function \( \hat{w} \) such as:

\[
\hat{w}(C, G_1, ..., G_N) = \hat{w}(I_1, ..., I_n)
\]  

(8)
HYPERELASTIC AND HYPOELASTIC MODELS FOR THE MESOSCOPIC ANALYSES OF COMPOSITE REINFORCEMENT DEFORMATION DURING FORMING

Number \( n \) depends on the number and type of arguments of the isotropic function under consideration. Finally, the strain energy function will be defined in the form (8). For sake of simplicity, it will be denoted \( w \) in the sequel. The symmetry group of a transversely isotropic material is characterised, in the initial configuration, by an unit vector \( \mathbf{M} \) which represents the preferred direction. This allows the definition of a structural tensor \( \mathbf{M} \mathbf{M} \mathbf{M} = \otimes \). Then, the strain energy potential becomes:

\[
I_1, I_2, I_3 \text{ are the standard invariants of } \mathbf{C} \text{ defined in (3). } II_4 \text{ and } II_5 \text{ are mixed invariants defined from the structural tensor } \mathbf{M} \text{ by:}
\]

\[
II_4 = \mathbf{C} : \mathbf{M} \quad \text{and} \quad II_5 = \mathbf{C}^2 : \mathbf{M}
\]

The second Piola-Kirchhoff stress tensor is then defined as:

\[
\mathbf{S} = 2 \frac{\partial w}{\partial \mathbf{C}} = 2 \left( \frac{\partial w}{\partial I_1} \frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial w}{\partial I_2} \frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial w}{\partial I_3} \frac{\partial I_3}{\partial \mathbf{C}} \right)
\]

\[
\frac{\partial w}{\partial II_4} \frac{\partial II_4}{\partial \mathbf{C}} + \frac{\partial w}{\partial II_5} \frac{\partial II_5}{\partial \mathbf{C}}
\]

\[
(11)
\]

2.5 Constitutive equation for a yarn

In this section, a hyperelastic model is proposed which describes the behaviour of the yarn in textile composite reinforcements. First, « physically based » invariants are defined which allow the description of the different deformation modes of the yarn introduced in section 2. Then, each mode will be associated a strain energy function from which the stresses in the material can be derived.

An approach based on a decomposition of the deformation gradient [9] is used to determine physically based invariants of the transformation for each deformation mode of the fibre bundle. An orthonormal basis \( \mathbf{B} = \{ \mathbf{M}, \mathbf{N}_1, \mathbf{N}_2 \} \) is defined which allows writing the components of the deformation gradient \( \mathbf{F} \) of transversely isotropic materials into the form:

\[
\begin{bmatrix}
0 & f_{m1} & f_{m2} \\
0 & f_{i} & 0 \\
0 & 0 & f_{22}
\end{bmatrix}
\]

(12)

As previously mentioned, \( \mathbf{M} \) is a unit vector which direction is the preferred direction of the material (i.e. the direction of fibres) in the initial configuration. A multiplicative split of the matrix (12) can be made in order to separate the parts corresponding to each deformation mode of the yarn:

\[
\begin{bmatrix}
\mathbf{F}_{\text{elong}} \mathbf{F}_{\text{comp}} \mathbf{F}_{\text{dist}} \mathbf{F}_{\text{sh}}
\end{bmatrix}
\]

\[
\times
\begin{bmatrix}
f_{m1} & f_{m2} & 1 \\
f_{m1} & f_{m2} & 1 \\
f_{m1} & f_{m2} & 1
\end{bmatrix}
\]

\[
(13)
\]

where \( \mathbf{F}_{\text{elong}}, \mathbf{F}_{\text{comp}}, \mathbf{F}_{\text{dist}} \) and \( \mathbf{F}_{\text{sh}} \) respectively describe the elongation of the yarn along the fibres direction, the compaction and distortion of the yarn in the transverse direction, and the longitudinal shear of the yarn. Each part of the decomposition can be characterized by a single quantity, except \( \mathbf{F}_{\text{sh}} \) which requires two quantities. Finally, five parameters are needed to describe the deformation with this approach:

\[
\alpha_{\text{elong}} = f_m, \quad \alpha_{\text{comp}} = \sqrt{f_{11} f_{22}},
\]

\[
\alpha_{\text{dist}} = \sqrt{f_{11} / f_{22}}, \quad \alpha_{\text{sh}} = \sqrt{f_{m1}^2 + f_{m2}^2}, \quad \tan(\gamma) = f_{m1} / f_{m2}
\]

(14)

This is consistent with the fact that five invariants are needed to write the strain energy potential (9) of hyperelastic transversely isotropic materials. With these notations, the components of the transformation gradient can be written as:
The quantity $\gamma$ is a measure of the coupling between the two shear modes, namely distortion and longitudinal shear. Under the uncoupling assumptions formulated before, the behaviour of the yarn is then assumed not to depend on this quantity. The standard invariants (9) can then be expressed as functions of the quantities (14). Inverting the obtained expressions leads to:

$$\alpha_{\text{long}} = \sqrt{I_4}, \quad \alpha_{\text{comp}} = \sqrt{\frac{I_5}{I_4}},$$

$$\alpha_{\text{dist}} = \sqrt{\frac{I_5 I_4 - I_5}{2 I_3 I_4} + \left(\frac{I_5 I_4 - I_5}{2 I_3 I_4}\right)^2} - 1, \quad (16)$$

$$\alpha_{\text{sh}} = \sqrt{\frac{I_5}{I_4} - 1}$$

Amongst the previous quantities, only $\alpha_{\text{sh}}$ vanishes when the material is not deformed (i.e. when $F = I_2$). After normalizing the other quantities, the following set of invariants is defined:

$$I_{\text{long}} = \frac{1}{2} \ln (I_4), \quad I_{\text{comp}} = \frac{1}{4} \ln \left(\frac{I_5}{I_4}\right),$$

$$I_{\text{dist}} = \frac{1}{2} \ln \left(\frac{I_5 I_4 - I_5}{2 I_3 I_4} + \left(\frac{I_5 I_4 - I_5}{2 I_3 I_4}\right)^2\right) - 1, \quad (17)$$

$$I_{\text{sh}} = \frac{I_5}{I_4} - 1$$

These invariants are used in the sequel to define the strain energy functions associated to each deformation mode:

$$w(C) = w(I_{\text{long}}, I_{\text{comp}}, I_{\text{dist}}, I_{\text{sh}}) \quad (18)$$

Strain energy functions associated to each deformation mode are then identified from simple experimental tests. Details are given in [2].

2.6 In-plane shear meso-scale deformation.

The previous hyperelastic constitutive law is used to perform a in-plane shear simulation and to compare the results to experimental values [9-12].

Fig. 3. Simulation of in-plane shear at meso-scale

The periodicity and boundary conditions imposed to the unit cell in this simulation of shear are similar to those described in [13]. The initial and deformed configurations of the unit cell are shown on Figure 3. Two criteria are used to compare the results: one which allows validating the overall efforts induced in the unit cell and one to validate its deformed geometry. In order to compare the results obtained by different measurement devices, or to compare the in-plane shear behaviour of different types of fabrics, a quantity must be defined which characterize the efforts in the unit cell. Different quantities have been introduced in previous papers.
In the sequel the torque \( C_s \) per unit initial area is used:

\[
C_s = F \frac{L_{frame}}{L_{fabric}^2} \frac{\cos(\gamma)}{2 \cos(\pi/4 - \gamma/2)}
\]  

(19)

with \( F \) the effort measured on the frame, \( L_{frame} \) the side of the frame, \( L_{fabric} \) the side of the square of fabric inside the frame and \( \gamma \) the angle variation between warp and weft directions (i.e. the shear angle).

On the other hand, an energy-based approach allows the calculation of this torque per unit initial area (19) from the simulation:

\[
C_s = \frac{\dot{W}}{S_u \dot{\gamma}}
\]  

(20)

where \( \dot{W} \) is the time-derivative of the external work, \( S_u \) the initial surface of the unit cell and \( \dot{\gamma} \) the time-derivative of the shear angle. The comparison of the torque obtained from experiments and simulations is presented in Fig. 4a. On this figure, the shear torques computed during the simulation and measured during the experiment are close. During the preparation of shear samples for experiments, a slight tension of the fabric is required to ensure a good positioning of the fabric in the picture frame. This tension induces stronger interactions between warp and weft direction, leading to a shift of the first part of picture frame experimental curves [12, 14]. A small equibiaxial strain (0.15%) has been used in the simulation to take into account the influence of the initial tension of the fabric.

3. Hypoelastic model for a fibre bundle

3.1. Rate constitutive equations

The rate constitutive equations (or hypo-elastic laws) are often used in finite element analyses at large strains [15-18]. User subroutines that can be implemented in codes such as ABAQUS to define the mechanical constitutive behaviour are written within this framework. A stress rate \( \dot{\sigma} \) is related to the strain rate \( D \) by a constitutive tensor \( \mathbf{C} \). To avoid rigid body rotations that can affect the stress state, the derivative \( \sigma^\nabla \), called the objective derivative, is the derivative for an observer who is fixed with respect to the material. Because this requirement is not uniquely defined there are several objective derivatives. In this work rotational objective derivatives correspond to a rotation tensor \( Q \) characterizing the rotation of the material.

The rate constitutive equation has the form:

\[
\sigma^\nabla = \mathbf{C} : D
\]  

(21)

with

\[
\sigma^\nabla = Q \left( \frac{d}{dt} \left( Q^T \sigma Q \right) \right) Q^T = \dot{\omega} + \sigma \Omega - \Omega \sigma
\]  

(22)

\( \Omega \) is the spin corresponding to \( Q \), i.e. \( \Omega = \dot{Q} Q^T \).

The most common objective derivatives are those of Green-Naghdi [19, 20] and Jaumann [21, 22]. In the case of the derivative of Green-Naghdi the rotation
$Q$ considered as that of the material is the rotation of the polar decomposition:

$$Q = R$$

(23)

which is derived from the decomposition $F = RU$ of the gradient tensor with $R$ the polar rotation and $U$ the right stretch tensor.

In the case of the derivative of Jaumann, $Q$ is the rotation of the corotational spinless frame:

$$Q = Rs$$

(24)

which is derived from the velocity gradient $\nabla v$ and the spin $Q_s = \frac{1}{2} (\nabla v - \nabla^T v) = \dot{R} \cdot R^T$.

These rate constitutive equations are written in the current configuration though hyperelastic constitutive equations are generally written in the initial configuration. Moreover hypo-elastic equations are a good choice for extension to history-dependent behaviour such as elasto-plasticity [23, 24].

During a finite element analysis the rate constitutive equation is used to update stresses once the displacement field and the corresponding strain field have been computed over the current time increment. Integrating Equation 1 over a time increment $\Delta t = t^{n+1} - t^n$ leads to the widely used formula of Hughes and Winget [15] for stress update:

$$\sigma_{i,j}^{n+1} = \sigma_{i,j}^n + [C_{i,j}^{n+1/2}]_{i,k} [\Delta \varepsilon_{k,l}^{n+1/2}]_{k,l}$$

with

$$[\Delta \varepsilon_{i,j}^{n+1/2}] = [D_{i,j}^{n+1/2}]_{i,j} [\Delta t]$$

(25)

The matrix $[S]_{i,j}$ denotes the components of the tensor $S$ in the basis $e_1 \otimes e_2 \otimes ... \otimes e_m$ at time $t^n$. Voigt notation is used: the components of a second-order tensor are arranged in a single column matrix. The basis vectors $e_i^n$ ($i = 1, 3$) come from the transportation at time $t^n$ of the initial basis vectors $e_i^0$ by the rotation $Q$ which defines the objective derivative given by Eq. 22. The frame $\{e_1^n, e_2^n, e_3^n\}$ denoted $\{e_i^n\}$ is called the rotated frame.

### 3.2 Textile composite reinforcement mechanical behaviour

Textile materials are made of fibres, which make their mechanical behaviour very specific. Relative sliding is possible between fibres (see Fig. 1a). The yarns are made of thousands of fibres and it is in general not possible to model each of them. The constitutive models that are introduced in the present paper are continuum models intended to describe the specific mechanical behaviour of the fibre bundle (Fig. 1). As a first step and to simplify the presentation a single fibre direction is considered as in Fig. 1. The equivalent macroscopic behaviour must take into account the fibrous nature of the material. The fibre direction stiffness dominates the overall stiffness. Consequently the constitutive tensor $C$ is oriented by $f_1$, the unit vector in the direction of the fibre at the considered point.

By default in commercial codes such as ABAQUS, large-strain analyses are based on Green-Naghdi (or Jaumann) rotated frame to update the Cauchy stress (equation (5)) and to update the orientation of the constitutive tensor $C$. These methods are not correct in the case of textile material because the polar rotation $R$ (or the rotation of the spinless frame $R_s$) is an average rotation of the material. After a large shear, $C$ is no longer oriented along the fibre direction [42]. Consequently this standard approach cannot be used for textile reinforcement forming analysis.

### 3.3 Objective derivative based on the fibre rotation

For fibrous material, an alternative to the standard Green-Naghdi or Jaumann objective derivatives is introduced [25]. The rate constitutive equation is based on the rotation of the fibre $f_1$:

$$\sigma^{\varphi} = C : D$$

(26)

with

$$\sigma^{\varphi} = \Phi \left( \frac{d}{dt} (\Phi^T \sigma \Phi) \right) \Phi^T$$
where $\Phi$ is the rotation of the fibre. It is shown in appendix B that the derivative $\sigma_r^{V_e}$ is objective. The stress update becomes:

$$[\sigma_r^{n+1}]_{i} = [\sigma_r^{n}]_{i} + [C^{n+1/2}]_{i} \Delta \epsilon_{r}^{n+1/2}$$  \hspace{1cm} (27)

From the transformation gradient $F_e$ the current fibre direction can be determined. Assuming that the initial position of the fibre is $f_1^0 = e_1^0$:

$$f_1 = \frac{F_e \cdot e_1^0}{\|F_e \cdot e_1^0\|}$$  \hspace{1cm} (28)

The other basis vectors $f_2$ and $f_3$ of the orthonormal frame $\{ f_i \}$ are obtained from the material transformation of $e_i^0$:

$$f_2 = \frac{F_e \cdot e_2^0 - (F_e \cdot e_1^0) f_1^0}{\|F_e \cdot e_2^0 - (F_e \cdot e_1^0) f_1^0\|}$$  \hspace{1cm} (29)

and $f_3 = f_2 \times f_1$.

The rotation $\Phi$ from $\{ e_i^0 \}$, the initial frame, to the fibre frame $\{ f_i \}$, is derived in the following way:

$$\Phi = f_1 \otimes e_1^0 = (f_1)_{i}^0 \otimes e_1^0 = (f_1, e_1^0) e_1^0 \otimes e_1^0$$  \hspace{1cm} (30)

The main interest of this approach is that the constitutive matrix in equation 27 appears in the frame of the fibre and consequently it is directly in its specific form corresponding to the textile material under consideration.

This constitutive matrix written in the fibre frame can be assumed constant in some cases. Generally it is not; the transverse behaviour of a fibrous yarn and the in-plane shear stiffness of a textile reinforcement are strongly strain state dependent.

When using a material user subroutine in a code such as ABAQUS, the strain increment $\Delta \epsilon$ is given at Gauss points in a frame which is not $\{ f_i \}$ but a standard frame. In the case of ABAQUS/Explicit it is Green-Naghdi’s frame $\{ e_i \}$ ($e_i = R e_i^0$). To use equation 27 it is necessary to calculate $[\Delta \epsilon]_i$ by a change of basis corresponding to the rotation $\Phi R^T$, later denoted as $\Theta$. In the same way when the stress update is performed with equation 27, it is necessary to return the stress components at $t^{n+1}$ in the code’s work frame using an inverse change of basis (corresponding to $\Theta^T$).

### 3.4. Numerical and experimental analysis of unit cells deformations

The hypo-elastic material model based on the fibre rotation, introduced above, is used to simulate the in-plane shear of a woven unit cell. There are two types of objectives for such a simulation. First, the macroscopic shear mechanical behaviour of the reinforcement can be determined, which is difficult experimentally. The picture frame and bias-extension tests that are used to this aim are difficult [11, 12]. The mesoscopic simulation can also be performed at the design stage of the reinforcement. Second, it provides local mesoscopic results such as yarn deformation and shape. These results are very important to perform flow simulation and to determine permeabilities of the reinforcement [26-28]. The fabric studied in this section is a carbon 2x2 twill. A geometric model is investigated to ensure its consistency, i.e. there are neither yarn penetrations nor unexpected voids [29]. The mesh of the yarns is mapped, which allows easily defining the fibre direction. Next, the choices of the unit cell and the boundary conditions have to render periodicity of the reinforcement. To this end the displacement field is split into a macroscopic average part and a local periodic part.

The material parameters are determined from a tensile test on a single yarn for the Young modulus and from an inverse method with an equi-biaxial tension test for the other parameters. The latter method is useful to determine the transverse behaviour because it features significant transverse crushing of warp yarns over weft yarns. At last, the friction coefficient it set to 0.2, which is a usual value for friction between carbon fibres [30-31].

The calculation was run in ABAQUS/Explicit using a VUMAT subroutine for the material model introduced in section 4. The deformed cell for a shear angle of $40^\circ$ is shown in Fig. 5. The present mesoscopic simulation enables the determination of the in-plane shear behaviour of the textile reinforcement: the curve of shear angle versus shear torque per unit initial surface is plotted in Fig. 6. The phenomenon of shear locking is well illustrated: from an angle of $30^\circ$ the macroscopic shear stiffness of the reinforcement increases significantly due to
the kinematics which leads to lateral contacts and crushing of the yarns. This result is compared to an experimental bias-extension test of this woven reinforcement. The difference between the shear stiffness before and after the locking angle is less important in the case of the experimental results than in the case of the computed ones. A possible explanation is that the numerical-model geometry is ideal (no defects) while the real geometry leads to a more random onset of contacts between the yarns throughout the fabric. On the whole, the agreement between the simulation and the experiments is fairly good.

A hyperelastic constitutive law has been presented to describe the mechanical behaviour of the yarns of textile composite reinforcements. Physically based invariants have been determined that highlight the different deformation modes identified in the yarn: elongation, compaction, distortion and longitudinal shear. A hyperelastic law has been developed based on these invariants: each deformation mode has been associated a strain energy function which allows describing the internal efforts in the yarn. This approach makes the physical meaning of the model appear clearly.

4. Conclusions

Hypo-elastic and Hyper-elastic approaches has been presented for large strain analysis of textile composite reinforcements. The hypo-elastic model is based on an objective derivative defined from the fibre rotation. The approach is simple and can be implemented in any commercial F.E. software. The proposed approach has been established in the case of a single fibre direction and applied to mesoscopic analyses.

Other mesoscopic simulations of the deformation of one or several unit cell are compared to experiments. For instance Fig. 7 and 8 show transverse compaction simulations with several plies with the same or different orientations.

References


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