A MECHANICAL MODEL FOR LAMINATED SHELLS WITH COHESIVE INTERFACES: VERIFICATION AND APPLICATIONS

F. Campi*, R. Massabò
Department of Civil, Chemical and Environmental Engineering, DICCA, University of Genova, via Montallegro 1, 16145, Genova, Italy
* Corresponding author (Francesca.Campi@unige.it)

Keywords: composite laminates, delaminations, cohesive interfaces, homogenization

1 Introduction
In the companion paper presented at this conference [1], a mechanical model has been presented to study laminated cylindrical shells with cohesive interfaces loaded dynamically. The model has been formulated within the framework of the discrete-layer approach and describes the shells as an assembly of sub-shells joined by cohesive interfaces. The cohesive interfaces describe different physical mechanisms: unfailed elastic interfaces, perfect adhesion of the layers, brittle and cohesive fracture and elastic contact along fracture surfaces. The model uses a homogenization technique to try and overcome the main drawbacks of classical discrete layer models, namely (i) the fact that they are computationally very expensive, since the number of unknown functions of the problem depends on the number $n$ of layers used to discretize the structure, and (ii) the fictitious compliance added to the system by the a-priori insertion of cohesive interfaces. In the model proposed in [1] continuity conditions are applied a-priori for shear and normal tractions at the interfaces along with the constitutive laws of the cohesive interfaces. The number of unknowns of the problem in a cylindrical shell is reduced to 6 (from $5 \times n!$) and becomes independent of the number of sub-shells. The model extends to shells with generally nonlinear cohesive interfaces the formulations originally proposed for intact shells in [2] and for shells with linearly elastic interfaces in [3].

In this paper, the model proposed in [1] is particularized to describe a shallow shell or a flat plate where all layers have the same elastic constants, e.g. unidirectionally reinforced, which deforms in cylindrical bending. Equilibrium equations and boundary conditions are derived for the quasi-static case. The model is then verified through a comparison with an exact 2D elasticity solution and with an improved third order model based on the theory proposed in [4].

2 First order homogenized model: assumptions and main equations
The model presented in the companion paper refers to a laminated cylindrical shell with arbitrary layup and loading and boundary conditions. Here the model is particularized to a shallow shell or a flat plate of thickness $h$, with equal layers, which is assumed to deform in cylindrical bending. A system of Cartesian coordinates $\alpha - \beta - z$ (or $x_1 - x_2 - x_3$) is introduced, with the axis $z$ normal to the reference surface and measured from it. The plate is subjected to a distributed load $p$ acting on the upper and lower surfaces. The displacement vector of an arbitrary point of the plate at the coordinates $(\alpha, \beta, z)$ is $u = \{v_\alpha, v_\beta, w\}^T$ where $v_\alpha = 0$, $v_\beta$ and $w$ are the components on the reference surface. The plate is modelled as an assemblage of $n$ sub-layers joined by $n-1$ cohesive interfaces, which define all actual and potential delamination surfaces in the structure (Fig. 1). The sub-layer $k$, where the index $k = 1,...,n$ is numbered from bottom to top, is defined by the coordinates $z^k$ and $z^{k+1}$ and has thickness $(^k)h$ (the $k$ superscript in brackets identifies affiliation with the sub-layer $k$). The sub-layers are linearly elastic, homogeneous and anisotropic with monoclinic symmetry with respect to their mid-surface; their principal material axes coincide with the geometrical axes of the structure $\alpha - \beta$ so as to satisfy the cylindrical bending conditions.
The equations of an arbitrary branch $i$ of the functions, where $iK_N^k$ and $iK_\beta^k$ are the interface stiffnesses in the transverse and longitudinal direction and $i'N^k$ and $i'\beta^k$ are cohesive tractions.

Describing the cohesive tractions as piecewise linear functions of the relative displacements allows to extend to plates with generally nonlinear cohesive interfaces the homogenization technique proposed by Librescu and Schmidt [5] for plates with purely elastic interfaces.

A purely elastic interface is described by a single branch with, $i'N^k = i'\beta^k = 0$, and the coefficients $K_N^k \to \infty$ and $K_\beta^k \to \infty$ define perfectly bonded interfaces. To describe perfectly brittle or cohesive fracture, the coefficients $iK_N^k$ and $iK_\beta^k$ in the initial branch of the laws, where $i'N^k = i'\beta^k = 0$, are chosen to be very high to minimize errors due to the introduction of fictitious compliant surfaces in the body.

The displacement field is assumed as a two length-scales field, given by the superposition of a global field (first two terms on the right hand side of the equations below) and local perturbation terms (last two terms on the right hand side):

$$v_\beta(\beta, z) = v_{\beta0}(\beta) + \varphi_\beta(\beta)z + \sum_{k=1}^{n-1} \Omega_{\beta}^k(\beta)(z-z^k)H^k + \sum_{k=1}^{n-1} \tilde{v}_\beta^k(\beta)H^k$$

$$w(\beta, z) = w_0(\beta) + \varphi_z(\beta)z + \sum_{k=1}^{n-1} \Omega_z^k(\beta)(z-z^k)H^k + \sum_{k=1}^{n-1} \tilde{w}_z^k(\beta)H^k$$

The terms on the right hand side of Eqs. (2) denote different contributions in the displacement representation: $v_{\beta0}$ and $w_0$ are the displacement components of a point on the reference surface of the plate and $\varphi_\beta$ the rotation of the normal to the reference surface (standard first order shear deformation plate theory contributions, $C_1^t$); $\varphi_z$ is
the deformation in the transverse normal direction, needed to capture, in the simplest way possible, the effect of transverse normal compressibility (this term is necessary in order to study delamination opening and elastic contact along the delamination surfaces); the third terms, with summations on the total number \( n-1 \) of interfaces and \( H^k = H(z-z^k) \), \( \{0, z < z^k; 1, z \geq z^k \} \), supply the zig-zag contributions, which are continuous in \( z \) but with jumps in the first derivatives at the interfaces (\( C_i^0 \)) and are necessary to satisfy continuity on normal and shear tractions at the interfaces in plate with nonhomogeneous layup; the fourth terms, with summations on the number of cohesive interfaces \( n-1 \), supply the contribution of the relative displacements (jumps) at the cohesive interfaces.

The strain components at the arbitrary coordinate \( z \) of the plate within sub-layer \( k \) are derived from the displacement field, Eq. (2):

\[
(4) \quad \varepsilon_{zz} = \varphi_z + \sum_{j=1}^{k+1} \Omega_i^j
\]

\[
(4) \quad \gamma_{\beta \beta} = \frac{\partial w_0}{\partial \beta} + \frac{\partial \varphi_{\beta}}{\partial \beta} z + \varphi_{\beta}
\]

\[
(4) \quad \gamma_{\mu \beta} = \frac{\partial w_0}{\partial \beta} + \frac{\partial \varphi_\mu}{\partial \beta} z + \frac{\partial \varphi_{\beta}}{\partial \beta} z^j + \sum_{j=1}^{k+1} \frac{\partial \Omega_i^j}{\partial \beta} (z-z^j) + \Omega_i^j + \frac{\partial \hat{\nu}_i^j}{\partial \beta}
\]

The unknown \( \Omega_i^j(\beta) \) and \( \Omega_i^k(\beta) \) in the displacement equations (2) are determined as functions of the generalized displacements of the reference surface and the displacement jumps by satisfying continuity conditions for shear and normal tractions across the laminate interfaces:

\[
(4) \quad \sigma_{\mu \beta}(z^j) = (k+1) \sigma_{\beta \beta}(z^j), \quad \sigma_{\beta \beta}(z^j) = (k+1) \sigma_{\beta \beta}(z^j).
\]

Where the stresses in the sub-shells are defined as functions of the strains in Eq. (3) through the linear-elastic constitutive equations, \( \sigma = C \varepsilon \).

In the derivation of the function \( \Omega_i^j(\beta) \) it is assumed that the effect of \( \varepsilon_{\mu \beta} \) on \( \sigma_{\mu \beta} \) is negligible with respect to that of \( \varepsilon_{\beta \beta} \), as suggested in [3]. This assumption allows the derivation of \( \Omega_i^j(\beta) \) directly from Eq. (4b) and \( \Omega_i^k(\beta) \), from Eq. (4a). The zig-zag functions are:

\[
(5) \quad \Omega_i^k(\beta) = 0 \\
(5) \quad \Omega_i^k(\beta) = -\frac{\partial \hat{\nu}_i^k}{\partial \beta}
\]

Once the functions \( \Omega_i^k(\beta) \) and \( \Omega_i^k(\beta) \) have been defined, the relative displacements at each cohesive interface, \( \hat{\nu}_i^k \) and \( \hat{v}_i^k \) for \( k = 1...n-1 \), are defined as functions of the transverse shear and normal strains at the interfaces, Eqs. (3) through the cohesive traction laws, Eqs. (1), and the constitutive equations of the sub-layers surrounding the interface. In order to perform this derivation, an initial status (piece \( i \) of the piecewise linear cohesive functions) is assumed for each interface of the plate; the status will then be updated through iteration at each loading step. The resulting displacement jumps are:

\[
(6) \quad \hat{\nu}_i^k = \frac{1}{K_N} \left[ \varphi_i C_{33} - t_i^k \right]
\]

(\( \text{the index } i \text{ which indicates the assumed branch of the cohesive traction laws has been removed in the equations above for sake of simplicity).}\)

Equations (5) and (6) can be inserted into equation (2) to obtain the homogenized displacement field in terms of \( v_{\mu \beta}, \varphi_\mu, w_0 \) and \( \varphi_z \) only:

\[
(7) \quad v_{\mu \beta} = v_{\mu \beta} + \varphi_z z + \left( \frac{\partial w_0}{\partial \beta} + \varphi_\mu \right) C_{55} \sum_{i=1}^{k+1} \frac{1}{K_i^\prime} \\
+ \frac{\partial \varphi_\mu}{\partial \beta} \sum_{i=1}^{k+1} \left[ C_{55} z^i K_N^i + C_{55} z^i K_N^i \right] - \sum_{i=1}^{k+1} \frac{t_i^k}{K_i^\prime}
\]

\[
(7) \quad w_\beta = w_\beta + \varphi_z \left[ z + C_{33} \sum_{i=1}^{k+1} \frac{1}{K_i^\prime} \right] - \sum_{i=1}^{k+1} \frac{t_i^k}{K_i^\prime}
\]
Equations (7) highlight that the displacement field is fully defined by the 4 displacement variables which define the global part of the displacement, and are underlined in the text, and by parameters which depend on the elastic constants of the material, the layup and the geometry (no line) and parameters depending on the properties of the interfaces through the assumed cohesive traction laws (curved line on top).

The equilibrium equations and boundary conditions are then derived in a weak form using the principle of virtual displacements:

\[
\int_V \sigma_c \delta w_j \, dV + \sum_{k=1}^{n} \left( t_{N}^k \delta \omega^k + t_{R}^k \delta \phi^k \right) dS - \int_S F_i^\beta \delta v_i \, dS - \int_I F_i^\theta \delta v_i \, dB = 0 \tag{8}
\]

where \( \delta w_j \) are the lower and upper surfaces of the plate and \( \delta \omega^k \) the lateral bounding surface of the body. The virtual displacements \( \delta v_i \), \( \delta \omega_0 \), \( \delta \phi_0 \) and \( \delta \phi_z \) are assumed to be independent and arbitrary and to satisfy compatibility conditions. The components of the external surface tractions acting along the bounding surfaces of the plate are \( F_i^{S^+} \) (top), \( F_i^{S^-} \) (bottom) and \( F_i^\theta \) (lateral). The second term in Eq. (8) defines the contribution to the elastic strain energy of the system due to the cohesive tractions \( t_{N}^k \) and \( t_{R}^k \) which is not accounted in the first term.

By substituting the homogenized displacement field, Eqs. (7), into the compatibility and constitutive equations, the strain and stress components in Eq. (8) are defined in terms of the global displacement variables. Then, applying Green’s theorem wherever possible, Eq. (8) leads to the approximate equilibrium and boundary conditions.

The equilibrium equations are presented here in terms of gross stress resultants and moments for the simplified case of slip only interfaces (namely, \( \phi_i = 0 \), \( K_N^k \rightarrow \infty \) or \( \dot{w}^k = 0 \), for \( k = 1,..n \)). They are:

\[
\frac{\partial Q_{\beta}}{\partial \beta} - \frac{\partial^2 M_{\beta}^I}{\partial \beta^2} + F_{z}^{S^+} + F_{z}^{S^-} - \frac{\partial F_{\beta}^{S^+}}{\partial \beta} - R_{S\beta}^n = 0 \tag{9}
\]

\[
\frac{\partial M_{\beta}^I}{\partial \beta} + \frac{\partial M_{\beta}^I}{\partial \beta} + Q_{\beta} - \check{C}_{55} \sum_{i=1}^{n-1} \frac{1}{K_{\beta}^i} + F_{\beta}^{S^+} (z^n + R_{S\beta}^n) + F_{\beta}^{S^-} z^0 = 0
\]

where the stiffness coefficients have been modified to account for negligible normal stresses in the \( z \) direction and plane strain conditions parallel to the axis \( \alpha \).

\[
I_i = \sum_{k=1}^{n} \int_{z=-1}^{1} \dot{z} F_{\beta}^k \, dz, \quad R_{S\beta}^k = \check{C}_{55} \sum_{i=1}^{n-1} \frac{1}{K_{\beta}^i}
\]

\[
I_i' = \sum_{k=1}^{n} \int_{z=-1}^{1} \dot{z} R_{S\beta}^k \, dz
\]

are parameters which describe the layup and the properties of the cohesive interfaces. The gross stress resultants and moments are:

\[
N_{\beta} = \sum_{k=1}^{n} \int_{z=-1}^{1} \left( \hat{\sigma}_{\beta} \right) \, dz, \quad M_{\beta} = \sum_{k=1}^{n} \int_{z=-1}^{1} \left( \hat{\sigma}_{\beta} \right) \, dz, \quad Q_{\beta} = \sum_{k=1}^{n} \int_{z=-1}^{1} \left( \hat{\sigma}_{\beta} \right) \, dz
\]

The static boundary conditions on \( \partial \epsilon \), at \( \beta = 0, L \), where stresses are prescribed are:

\[
\begin{bmatrix} N_{\beta} n_{\beta} - \check{N}_{\beta}^n = 0 \end{bmatrix}_{0,L}
\]

\[
\begin{bmatrix} Q_{\beta} n_{\beta} - \check{Q}_{\beta}^n + \check{C}_{55} \sum_{i=1}^{n-1} \frac{1}{K_{\beta}^i} n_{\beta} - \check{R}_{\beta}^n \end{bmatrix}_{0,L}
\]

\[
\begin{bmatrix} M_{\beta} n_{\beta} + \check{M}_{\beta} n_{\beta} - \check{M}_{\beta}^n = 0 \end{bmatrix}_{0,L}
\]

\[
\begin{bmatrix} M_{\beta} n_{\beta} - \check{M}_{\beta}^n = 0 \end{bmatrix}_{0,L}
\]

where the resultant forces and moments are defined as:
\[
\tilde{N}_\beta = \sum_{k=1}^{n} \int^{(i)} F^\beta_n \, dz, \quad \tilde{N}_n^\beta = \sum_{k=1}^{n} \int^{(i)} F^\beta_n \, dz,
\]
\[
\tilde{M}_\beta = \sum_{k=1}^{n} \int^{(i)} F^\beta_n \, dz, \quad \tilde{M}_n^\beta = \sum_{k=1}^{n} \int^{(i)} F^\beta_n R^\beta_{s\beta} \, dz.
\]

(12)

The geometric boundary conditions on \( v_0 \), where displacements are prescribed, are \( v_{\beta 0} = \tilde{v}_{\beta 0} \), \( w_0 = \tilde{w}_0 \), \( \varphi_\beta = \tilde{\varphi}_\beta \) and \( \partial w_0 / \partial \beta = \partial \tilde{w}_0 / \partial \beta \), where the tilde sign identifies prescribed values.

For a plate with perfectly bonded interfaces, namely when \( K^\beta_\infty = \infty \) at all interfaces, all terms with the upper curved line cancel and the equilibrium equations reduce to those of a homogeneous plate.

The equilibrium equations (9) and the static boundary conditions (11) can be written in terms of displacements introducing the homogenized displacement field of Eqs. (7) in the gross stress resultants and moments of Eqs. (10). The equilibrium equations become:

\[
\frac{\partial^2 w_0}{\partial \beta^2} \tilde{c}_{22} I_0 + \frac{\partial^2 \nu_{\beta 0}}{\partial \beta^2} \tilde{c}_{22} \tilde{I}_0
\]
\[
+ \frac{\partial^2 \varphi_\beta}{\partial \beta^2} \tilde{c}_{22} \left( I_1 + I_0 \right) + F_{\beta}^s + F_{\beta}^s = 0
\]
\[
\left( \frac{\partial^2 w_0}{\partial \beta^2} + \frac{\partial \varphi_\beta}{\partial \beta} \right) \tilde{c}_{22} K^2 I_0 - \frac{\partial^4 w_0}{\partial \beta^2} \tilde{c}_{22} \sum_{k=1}^{n} \left( R^\beta_{s\beta} \right)^2 \left( \tilde{h} \right) I_1
\]
\[
- \frac{\partial^3 v_{\beta 0}}{\partial \beta^2} \tilde{c}_{22} \tilde{I}_0 + \frac{\partial^3 \varphi_\beta}{\partial \beta^2} \tilde{c}_{22} \left( \sum_{k=1}^{n} \left( R^\beta_{s\beta} \right)^2 \left( \tilde{h} \right) I_1 \right)
\]
\[
+ F_{\beta}^s + F_{\beta}^s - \frac{\partial F_{\beta}^s}{\partial \beta} R^\beta_{s\beta} = 0
\]
\[
\frac{\partial^3 w_0}{\partial \beta^2} \tilde{c}_{22} \left[ I_1 + \sum_{k=1}^{n} \left( R^\beta_{s\beta} \right)^2 \left( \tilde{h} \right) \right] + \frac{\partial^3 \nu_{\beta 0}}{\partial \beta^2} \tilde{c}_{22} \left( I_1 + I_0 \right)
\]
\[
+ \frac{\partial^2 \varphi_\beta}{\partial \beta^2} \tilde{c}_{22} \left[ I_2 + 2 I_1 + \sum_{k=1}^{n} \left( R^\beta_{s\beta} \right)^2 \left( \tilde{h} \right) \right]
\]
\[
- \left( \frac{\partial \nu_{\beta 0}}{\partial \beta} + \varphi_\beta \right) \tilde{c}_{55} K^2 I_0 - \tilde{c}_{55} \sum_{k=1}^{n} t^\beta_{s\beta}
\]
\[
+ F_{\beta}^s \left( z^0 + R^\beta_{s\beta} \right) + F_{\beta}^s \tilde{z}^0 = 0
\]

(13)

\[
\frac{\partial^2 w_0}{\partial \beta^2} \tilde{c}_{22} I_0 + \frac{\partial \varphi_\beta}{\partial \beta} \tilde{c}_{22} \left( I_1 + I_0 \right)
\]
\[
+ \frac{\partial^2 \varphi_\beta}{\partial \beta^2} \tilde{c}_{22} \tilde{I}_0 \]
\[
- \frac{\partial^2 \nu_{\beta 0}}{\partial \beta^2} \tilde{c}_{22} \tilde{I}_0 \]
\[
+ \frac{\partial^2 \varphi_\beta}{\partial \beta^2} \tilde{c}_{22} \tilde{I}_1 \]
\[
+ \frac{\partial^2 \nu_{\beta 0}}{\partial \beta^2} \tilde{c}_{22} \tilde{I}_1 \]
\[
+ \left( \frac{\partial^2 \nu_{\beta 0}}{\partial \beta^2} + \frac{\partial \varphi_\beta}{\partial \beta} \right) \tilde{c}_{22} \left( I_1 + \sum_{k=1}^{n} \left( R^\beta_{s\beta} \right)^2 \left( \tilde{h} \right) \right)
\]
\[
\left. \right|_{n_\beta - \tilde{M}_\beta = 0}^{0 \rightarrow L}
\]

(14)

3 Third order homogenized model: assumptions and main equations

In this section the third order model formulated in [4] is particularized to the case of a unidirectionally reinforced flat plate deforming in cylindrical bending with slip only cohesive interfaces.

Following [4], the displacement field is defined as:

\[
v_\beta (\beta, z) = v_{\beta 0} (\beta) + \varphi_\beta (\beta) z + \theta (\beta) z^2 + \psi (\beta) z^3
\]
\[
+ \sum_{k=1}^{n} \Omega^k (\beta) (z - z^k) H^k + \sum_{k=1}^{n} \tilde{\nu}^k (\beta) H^k
\]
\[
w(\beta) = w_0 (\beta)
\]

and the static boundary conditions are:
The transverse shear strain at the arbitrary coordinate \( z \) of the plate within sub-layer \( k \) is:

\[(4) \gamma_{\beta z} = \frac{\partial w_0}{\partial \beta} + \varphi_{\beta} + 2\varphi z + 3\psi z^2 + \sum_{j=1}^{4} \Omega_j (\beta). \tag{16}\]

Assuming a linearly-elastic constitutive behavior in the layers, the number of unknown functions is reduced by imposing the conditions of zero shear stresses at the upper and bottom surfaces of the plate:

\[
\varphi_{\beta}(\beta) = A f z L z, \quad \psi(\beta) = -A f z L z, \quad \varphi, \psi \quad \text{subjected to a quasi-stress}\]

The resulting homogenized displacement field is then obtained in terms of \( v_{\beta 0}, w_0 \) and \( \varphi \) :

\[
(4) v = v_{\beta 0} - \frac{\partial w_0}{\partial \beta} z + \varphi \left( z^2 - \frac{2z^2}{3} h \right) + \sum_{i=1}^{4} \frac{1}{K_i} \left( \tilde{C}_{\alpha \beta} \varphi \left[ 2z^2 - 2 \left( z^2 \right) / h \right] - t_{\beta i} \right) \tag{19}\]

The exact elasticity solution obtained by Pagano [6] for intact multilayered plates and extended in [7] to plates with linearly elastic interfaces, is extended here to plates with linear non proportional interface laws.

The solution is presented for a unidirectionally reinforced plate simply supported at the edges at \( \beta = 0 \) and \( \beta = L \) and subjected to a quasi-static distributed transverse load \( q = q_0 \sin \left( \frac{\pi \beta}{L} \right) \). The plate is in cylindrical bending and has two layers and a single mid-surface cohesive interface. Local coordinates \( \beta, z_4 \) are introduced with the origin at the 4th layer mid-plane and the axis \( \beta \) parallel to \( \beta \).

The solution is based on a stress function approach, where the stress function for the \( k \)th layer is assumed as:

\[
(4) \Phi(\beta, z_k) = f_k(z_k) \sin \left( \frac{\pi \beta}{L} \right) \tag{20}\]

with \( k = 1, 2 \). The function satisfies equilibrium in the layer. Through the constitutive equations for cylindrical bending in an orthotropic material with principal axes \( \beta, z_k \), compatibility yields:

\[
\tilde{A}_{22} \frac{\partial^2 f_k(z_k)}{\partial z_k^2} \left( \tilde{A}_{33} + 2\tilde{A}_{23} \right) \left( \frac{\pi}{L} \right)^2 \frac{\partial^2 f_k(z_k)}{\partial z_k^2} + \tilde{A}_{33} \frac{\partial f_k(z_k)}{\partial z_k} \left( \frac{\pi}{L} \right) = 0 \tag{21}\]

where the tilde indicates reduced compliance coefficients to account for plane strain conditions orthogonal to the plane \( \beta, z_k \).

The solution of the differential equation yields the following displacement field:

\[
(4) v = \frac{\cos \left( \frac{\pi \beta}{L} \right)}{\pi/L} \sum_{j=1}^{4} C_j \left[ \tilde{A}_{23} \left( \frac{\pi}{L} \right)^2 - \tilde{A}_{22} \lambda_j^2 \right] e^{\lambda_j \beta} \tag{22}\]

\[
(4) w = \frac{\sin \left( \frac{\pi \beta}{L} \right)}{\pi/L} \sum_{j=1}^{4} D_j \left[ \tilde{A}_{33} \lambda_j - \tilde{A}_{23} \left( \frac{\pi}{L} \right)^2 \right] e^{\lambda_j \beta} \tag{23}\]

where

4 Exact 2D elasticity solution
\[ \lambda_j = \pm \frac{\pi}{L} \sqrt{\frac{\hat{A}_{34} + 2\hat{A}_{14} \pm \left( \hat{A}_{34} + 2\hat{A}_{14} \right)^2 - 4\hat{A}_{22}\hat{A}_{33}}{2\hat{A}_{22}}} . \]

The eight arbitrary constants, \( C_j \) and \( D_j \) for \( j = 1 \ldots 4 \), are determined by imposing boundary and continuity conditions:

\[
\begin{align*}
(2) \sigma_{zz}(\beta, z_2 = h/4) &= -q_0 \sin \left( \frac{\pi \beta}{L} \right) \\
(1) \sigma_{zz}(\beta, z_1 = -h/4) &= 0 \\
(1) \sigma_{\beta z}(\beta, z_i = -h/4) &= (2) \sigma_{\beta z}(\beta, z_2 = h/4) = 0 \\
(1) \sigma_{\beta z}(\beta, z_i = h/4) &= (2) \sigma_{\beta z}(\beta, z_2 = h/4) \\
(1) \sigma_{\beta z}(\beta, z_1 = h/4) &= (2) \sigma_{\beta z}(\beta, z_2 = -h/4) \\
\end{align*}
\]

and the interface laws:

\[
\begin{align*}
(1) \sigma_{\beta z}(\beta, z_i = h/4) &= K_{\beta} \tilde{v}_i \tilde{t}_{\beta} \\
(1) \sigma_{\beta z}(\beta, z_i = h/4) &= K_{\beta} \tilde{w} \tilde{t}_N .
\end{align*}
\]

5 Model verification and applications

A unidirectionally reinforced plate with two layers, \( n = 2 \), and a single, mid-surface, slip only, cohesive interface is considered (Fig. 2). The plate is assumed to be in cylindrical bending conditions, simply supported at the edges at \( \beta = 0 \) and \( \beta = L \) and subjected to a quasi-static distributed transverse load \( q = q_0 \sin \left( \frac{\pi \beta}{L} \right) \).

\[ E_3 = E_2 = 0.07E_2 \quad , \quad G_{23} = E_2 / 21 \quad , \quad \nu_{12} = 0.02 \quad , \quad \nu_{32} = 0.02 \quad , \quad \nu_{13} = 0.54 . \]

The solutions of the first and third order homogenized models are compared with the exact 2D elasticity solution. In Figure 3 the dimensionless deflection \( w_c \hat{C}_{22}/(q_0 h) \) is shown as a function of the dimensionless interfacial compliance, \( \hat{C}_{22}/K_{\beta}h \), where \( \hat{C}_{22} = E_2/(1 - \nu_{12}\nu_{21}) \).

The diagram shows that for \( K_{\beta} \to \infty \) when \( L/h \) is sufficiently large as in the example shown, all curves tend to the solution of a shear deformable intact plate, which is well described by the First Order Shear Deformation theory. For small values of \( K_{\beta} \) the models predict different trends. In the limiting case of a plate with a fully imperfect interface ( \( K_{\beta} = 0 \) ) the elasticity 2D solution (thick solid curve) tends to the solution of two single shear deformable plates of thickness \( h/2 \) free to slide over each other. On the other hand, the first and third order solutions (thin and dashed curves) tend to the solution of two single Kirchhoff–Love plates of thickness \( h/2 \) free to slide over each other. This is a consequence of the continuity conditions imposed on the transverse stresses at the interface: assuming \( K_{\beta} = 0 \) at the interface leads to vanishing shear stresses in the layers. When the solution tends to this limit, on decreasing the interface stiffness, the homogenized models then accurately describe the response of plates where shear deformations are negligible (e.g. where \( L/h \) end/or the shear stiffness are large enough). Work is in progress to verify if the model can be used to describe plates where these assumptions are not satisfied by introducing appropriate shear factors.

For intermediate values of \( K_{\beta} \) the first and third order solutions present an unrealistic trend, which differs from the exact elasticity solution and shows an initial drop where the displacements become larger than the limiting value. This behavior seems to be a consequence of the condition in Eq. (11b) (and of a similar condition for the third order model), which lets the shear force in the plate to differ from the resultant of the vertical forces reduced at the cross section when \( K_{\beta} \neq \infty \). The same condition is satisfied by a zero shear force.

\[ \begin{align*}
\text{Fig. 2. Flat plate under cylindrical bending conditions with mid-plane cohesive interface.}
\end{align*} \]
when \( K_\beta = 0 \) (see previous paragraph). Work is in progress to verify if the problem can be controlled by a properly defined shear factor. The diagram in Figure 4 shows the dimensionless interfacial shear traction at \( \beta = 0 \) as a function of the dimensionless interfacial stiffness. In the limiting case of an intact plate \( (K_\beta = \infty) \) both the first (thin solid curve) and the third (dashed curve) order models approach the 2D elasticity solution (thick solid curve). All curves start from a zero interfacial shear traction in the limiting case of a fully imperfect interface, \( K_\beta = 0 \). For larger values of \( K_\beta \) the 2D elasticity solution monotonically decreases towards the horizontal asymptote given by the solution of the intact plate. The first and third order models also approach the limiting solution; however, they exhibit an initial drop which goes well below the asymptotic limit. The curve corresponding to the first order model has been obtained by multiplying by 3/2 the constant shear stresses of the solution of the problem, in order to approximately account for the actual distribution of the stresses. The dotted curve depicts the interfacial tractions obtained without the correction.

Figure 5 shows the dimensionless interfacial shear tractions at \( \beta = 0 \) as calculated a posteriori from equilibrium conditions in the first (thin solid curve) and third (dashed curve) order models. The diagram shows that the calculation based on equilibrium consideration removes the problems highlighted in the previous diagram.

Figure 6 shows the transverse shear stresses in the thickness of the plate, calculated a posteriori from equilibrium, for three different values of the dimensionless interfacial stiffness \( \bar{K}_\beta = K_\beta h/2\bar{E}_{22} \). The predictions of the first and third order model accurately describes the exact solution but for the intermediate values of the interfacial stiffness.

6 Conclusions

The model proposed in the companion paper [1] is particularized to describe a unidirectionally reinforced shallow shell or flat plate, deforming in cylindrical bending conditions. The model is compared with a third order model and an exact 2D elasticity solution (extensions, respectively, of the models proposed in [4] and [6], to account for linear non proportional interface cohesive laws). Some limitations of the model are highlighted by the comparison. Work is in progress to verify improvements in the solutions through the use of shear factors.

Acknowledgements: work supported by U.S. Office of Naval Research, no. N00014-05-1-0098, administered by Dr. Y.D.S. Rajapakse, and Italian MIUR, Prin09 no. 2009XWLFKW.

References

Fig. 3. Mid-span deflection of the mid-plane of the unidirectionally reinforced plate with a single elastic interface as a function of the interfacial compliance coefficient. Comparison of the solutions of the first and third order models and the exact 2D elasticity solution.

Fig. 4. Interfacial shear tractions at $\beta = 0$ in the unidirectionally reinforced plate with a single mid-plane elastic interface as a function of the interfacial stiffness. Comparison of the solutions of the first and third order models and the exact 2D elasticity solution.

Fig. 5. Interfacial shear tractions at $\beta = 0$ in the unidirectionally reinforced plate with a single mid-plane elastic interface as a function of the interfacial stiffness. Solutions of the first and third order models have been derived a posteriori through equilibrium.

Fig. 6. Transverse shear stresses at the cross section at $\beta = 0$ in the unidirectionally reinforced plate with a single mid-plane elastic interface for $-h/2 \leq z \leq 0$ and for different values of the interfacial stiffness. Comparison of the solutions obtained a posteriori from equilibrium in the homogenized first and third order models and the exact 2D elasticity solution.