1 Introduction

Structural failure is a phenomenon that has to be anticipated and possibly avoided by designers and engineers. One of its most common features is the generation of discontinuities in the material such as cracks, voids, material separation ... At an early stage of the damage process such discontinuities are present but do not compromise stability and safety therefore their presence may be interpreted as a premonition which can be recognised before failure takes place. Therefore it would be of the highest importance to be able to fully describe the damage process affecting structural components in order to evaluate their life expectancy for a safe but economically sound utilisation.

Peridynamics is an integral theory of continuum mechanics that can easily describe crack propagation. It is the base of computational methods that simulate dynamic crack propagation such as those due to impacts or explosions [1-8].

Since cracks in solids represent discontinuities in the material, the classical theory of solid mechanics, based on partial differential equations, encounters many difficulties when applied to cracked bodies. The use of Peridynamics can overcome these problems, because it does not require spatial derivatives to be evaluated.

When dealing with discontinuities, the classical finite element method has to be equipped with ad hoc developed tools such as interface elements [9] or x-fem [10]. However interface elements can be used if the path of the crack is known a-priori and the x-fem method does not find easy application in 3-d cases. Peridynamics is more general in the sense that the crack is free to appear in every part of the structure, following only physical and geometrical constraints and does not seem to be more difficult to apply in 3-d than in the planar cases. From an algorithmic point of view the need to describe crack propagation requires additional features in the finite elements crossed by the crack paths. In the peridynamic approach instead all grid points are equipped with the same ‘capabilities’ and no additional complexity is required in order to simulate crack propagation. On the contrary the cracked grid is obtained from the original undamaged grid simply by removing the broken bonds so that the cracked grid could be seen as a sub-set of the original grid.

The original formulation of the computational methods based on the peridynamics theory regarded dynamic problems which were solved with an explicit time integration scheme. The researchers who proposed Peridynamics adapted their algorithms to the static case by introducing in their computation a considerable amount of numerical damping [3]. The present paper proposes a simpler static version of the peridynamic formulation which has been implemented as an implicit scheme based on a Newton-Raphson algorithm. It is applied to the nonlinear analysis of static cases of crack propagation in deforming bodies.

The paper gives a brief introduction to the fundamentals of our static implementation of the peridynamics theory and to the material nonlinearities used to simulate damage and crack propagation. Results are presented at the end for three structural configurations.

2 Fundamental concepts

2.1 Static approach of Peridynamics

Bond-based Peridynamics’ main assumption is that the body is composed of material points which are linked to any other point within a finite distance named horizon. These connections are called bonds.
All equations that govern the motion of the points of the structure are written in the initial reference configuration, and they assume the following form:

$$\rho \ddot{u}_i = \int_H f(u(x_j,t) - u(x_i,t), x_j - x_i) dV_j + b(x_i,t)$$  \hspace{1cm} (1)

where $H_i$ is the neighbourhood of point $x_i$ limited by the horizon length $\delta$, $x_j$ represents any point within the neighbourhood of point $x_i$, $u$ is the displacement vector field, $b$ is a body density force vector and $f$ is called pairwise force function (with units of force per unit volume squared), that represents the force that point $x_j$ exerts on point $x_i$.

This force $f$ acts along the line connecting the two points, therefore it does not produce any moment on them. Under this assumption, and naming

$$\xi = x_j - x_i$$

$$\eta = u(x_j,t) - u(x_i,t)$$

the relative position and displacement respectively, the stretch $s$ of the bond can be expressed by the following formula:

$$s = \frac{|\eta + \xi|}{|\xi|}$$  \hspace{1cm} (3)

In the linear elastic range, $s$ represents the ratio of the norm of the pairwise force function $f$ to the bond stiffness $c$, therefore it is possible to express $f$ in the following form:

$$f = c \cdot s \cdot \mu(\xi) \cdot \frac{\eta + \xi}{|\eta + \xi|}$$  \hspace{1cm} (4)

where $\mu(\xi)$ is a history-dependent parameter that depends on the adopted constitutive law of the material, and can take into account the differences between brittle, elastic and elastic-plastic behaviours.

The numerical approximation of the Peridynamic equation starts with the division of the structure into nodes, each one associated to a certain volume, that summed up cover the entire body.

Therefore, the discretized form of (1) is given as follows:

$$\rho \ddot{u}_i^n = \sum_j f(u_j^n - u_i^n, x_j - x_i) \Delta V_j + b_i^n$$  \hspace{1cm} (5)

where $i$ is the index of the central node, also called source node, and it is fixed in eq.(5), $j$ is the family node index and represents all nodes within the horizon of node $i$, $n$ is the time step number, $\Delta V_j$ is the discretized volume associated to node $j$ (see Figure 1).

For the purposes of this paper, the time dependence is removed from equation (5), and the first term is set equal to zero so that static equilibrium is expressed as follows:

$$\sum_j f(u_j^n - u_i^n, x_j - x_i) \Delta V_j + b_i^n = 0$$  \hspace{1cm} (6)

In equation (6) $n$ indicates the load increment, since the problem can be studied as a sequence of static analyses with incremental external loading in terms of imposed forces and/or displacements.

2.2 Non-linear constitutive behaviour

In this paper, the material adopted to model each bond is characterised by a bilinear relationship between the pairwise force function $f$ and the bond stretch $s$, in order to model progressive bond damaging phenomena. This relationship is shown in Fig. 2(b), whereas Fig. 2(a) represents the constitutive law of the so called prototype brittle material (PMB) [6]:

$$f = \begin{cases} c \cdot s, & s \leq s_p \\ c \cdot s_p, & s > s_p \end{cases}$$

In equation (6) $n$ indicates the load increment, since the problem can be studied as a sequence of static analyses with incremental external loading in terms of imposed forces and/or displacements.

$$\sum_j f(u_j^n - u_i^n, x_j - x_i) \Delta V_j + b_i^n = 0$$  \hspace{1cm} (6)

In equation (6) $n$ indicates the load increment, since the problem can be studied as a sequence of static analyses with incremental external loading in terms of imposed forces and/or displacements.
\( s_0 \) is the maximum bond stretch for which the material is linear elastic, \( s_c \) is the bond stretch at failure, if \( s_{\text{max}} \), the maximum value of the stretch during the deformation history of the bond, is between \( s_0 \) and \( s_c \) the bond is weakening.

Moreover, a bond in compression never fails and its stiffness remains equal to the initial one whenever the stretch becomes negative. Similar constitutive laws have often been adopted in damage mechanics [9] and can be expressed in mathematical form as follows:

\[
G(s) = \begin{cases} 
  \c s & s \leq s_0 \\
  c \left(1 - \frac{s_{\text{max}} - s_0}{s_{\text{max}} - s_{\text{c}}} \right) s & s < s_{\text{max}} \text{ and } s_{\text{max}} > s_0 \\
  \frac{c}{s} \left(1 - \frac{s_{\text{c}} - s_0}{s_{\text{c}} - s_{\text{max}}} \right) s & s = s_{\text{max}} \text{ if } s_{\text{max}} > s_0 \\
  0 & s \geq s_{\text{c}}
\end{cases}
\]  

(7)

The area of the triangle under the bilinear constitutive law is related to the critical energy release rate of the material \( G_c \) for mode I fracture. The failure stretch \( s_c \), which defines the condition of failure of the bond, is clearly related with \( G_c \). This value is estimated by evaluating the energy required to have the complete separation of the fracture surfaces during crack propagation [1]. In the peridynamic approach this requires the evaluation of the following integral, for the case of a prototype brittle material:

\[
G_c = \frac{\delta}{2\pi} \int_0^{2\pi} d\phi \int_0^\infty d\xi \xi 2 \int_0^{\delta} \xi \frac{d^2G_c}{d\xi^2} \frac{\xi^2}{2} \sin(\phi) d\xi
\]  

(8)

Moreover, this integral must be estimated in a domain equal to one horizon sphere \([4]\). It can be shown that the limit stretch \( s_c \) in the case of a 3D prototype brittle material [1] is given by:

\[
s_c = \frac{5G_c}{6E\delta}
\]  

(9)

and for the 2D cases it takes the following expression:

\[
s_c = \frac{4\pi G_c}{9E\delta} \quad \text{plane stress}
\]  

(10)

\[
s_c = \frac{5\pi G_c}{12E\delta} \quad \text{plane strain}
\]  

Similarly, the integral of eq. (8) can be evaluated using the bilinear force function of Fig. 2(b).

In fact, naming \( G_0 \) the energy required to start the damaging process (i.e. the stretch reaches the value \( s_0 \)), it is possible to estimate \( s_0 \):

\[
s_0 = \sqrt{\frac{4\pi G_0}{9E\delta}} \quad \text{plane stress}
\]  

(11)

\[
s_0 = \sqrt{\frac{5\pi G_0}{12E\delta}} \quad \text{plane strain}
\]

2.3 On the damage index

Cohesive zone models have often been used to model crack propagation [9]. In the cohesive zone model literature the weakening of the bonds represented by the softening branch shown in Fig. 2 is often called damage. In the peridynamic literature [4] another concept of damage is often used, which refers to the portion of broken bonds in a material point. The damage index \( \phi \) in a node of the grid can be defined as the ratio of the number of broken bonds and the total number of bonds initially connected with that node. A crack usually separates nodes which end up being on the two different sides of the crack itself. Fig. 3 is a sketch which should help understanding the definition of the damage index, from the point of view of Peridynamics as a theory of continuum (i.e. not considering the numerical discretisation adopted in the numerical examples). In case (a) the crack separates the material point \( P \) from 50% of its neighbours, in case (b) from three quarters and in case (c) from one quarter only. Fig. 3 clarifies that with such a definition of damage fully developed cracks can correspond to levels of damage much smaller than 1. Moreover in the case of a discrete grid, such as those used in computer models, in which every node is connected to a finite, and usually small, number of other nodes, damage levels can assume a discrete set
of values, depending on the ratio $m = \frac{\delta}{\Delta x}$ which is equal to 3 in the present paper. Fig. 4 shows that a straight crack parallel to the one of the two main grid directions generates a damage index close to 0.4.

![Fig. 4: In a discrete grid a crack path intersects a finite number of bonds so that the damage index $\varphi$ can assume only a finite number of discretised values.](image)

### 3 Computational approach

The small displacement assumption is made so that no geometrical non-linearity is involved ($\eta = 0$ in equation 4). However, since the constitutive law of the bonds is bilinear and the bonds can break the stiffness of the structure is not constant but history dependent, and therefore the overall problem is nonlinear. In fact, when some bonds overcome the linear elastic limit and become damaged, the entire structure’s stiffness decreases and needs to be updated. For this reason, an iterative procedure based on the Newton-Raphson (N-R) method can be used to solve numerically the problem [11, 12].

A full N-R method is adopted, that updates the stiffness matrix at each iteration. Using the conventional finite elements terminology, the equilibrium relationship between external and internal forces of the structure takes the following form:

$$[K]u = F$$  \quad (12)

where $[K]$ is the global stiffness matrix of the structure, $u$ is the vector of nodal displacements (containing two degrees of freedom per node, because of the two-dimensionality of the problem) and $F$ is the external force vector. Since the problem is nonlinear, there is a direct dependence of the structure stiffness matrix on the displacements vector:

$$[K] = [K(u)]$$  \quad (13)

The out of balance vector $g$ needs to be evaluated, in order to check if the force equilibrium is verified, namely if its values are all zero (within a given tolerance) for a given value of the displacement vector $u_0$:

$$g = F_{\text{EXTERNAL}} - F_{\text{INTERNAL}} = [K]u - F$$  \quad (14)

If this function is not the zero vector, (according to the Newton Raphson method [11]) it is necessary to calculate the following derivative in $u_0$:

$$\frac{\partial g}{\partial u} = \partial \left( [[K]u - F] \right) = [K] + \frac{\partial |K|}{\partial u}$$  \quad (15)

The term $\frac{\partial |K|}{\partial u}$ represents the tangent to the equilibrium curve $g = g(u)$ in the point $u = u_0$, and it is called tangent stiffness matrix. The contribution to the global stiffness matrix from a single bond depends on the current value of the stretch $s$ and on the previous stretch history of the bond.

### 4 Results

A grid size sensitivity study has been carried out [12] both for linear elastic problems and for nonlinear crack propagation problems. The study shows that the solution to a static problem provided by the peridynamic based computational method converges to the exact solution if the grid size is uniformly reduced. Moreover it seems that the convergence rate of the peridynamic solution is slower than that of the finite element solution. In the examples solved in [12] the fem results obtained with a model containing a given number of degrees of freedom are usually closer to the exact solution than the peridynamic results obtained with the same number of degrees of freedom. The peridynamic solution is affected by particular problems, related to the so-called skin effect [4] and to the specific way external forces and boundary conditions are applied to the grid of the peridynamic model, which reduce the convergence rate with the grid size reduction.

In the examples presented in the paper an initial crack is defined as a line across a portion of the structure which is not intersected by any active
bond. Initial cracks are obtained by simply removing from an otherwise uniform grid and bond distribution all bonds which would intersect the initial crack. Peridynamics based numerical methods do not need initial cracks, since a crack can start anywhere in the structure a given criterion is met. However initial cracks are used in the paper in order to provide the reader with an understanding of where the crack should start from.

Several examples of static crack propagation in plate-structures have been simulated [12], three of which are shown in the figures of the present section.

**Example 1**

A double cantilever beam test on a pre-cracked specimen has been simulated with the developed method. In Fig. 5 the geometry and boundary conditions are shown: the external load is an imposed displacement the amplitude of which varies linearly along the edge of the plate.

The dimensions of the specimen are: $W=2.5$ mm, $H=5.1$ mm, $t=0.25$ mm (thickness) and the crack, which lies on the horizontal symmetry axis of the plate, has an initial length $a=1.2$ mm. The chosen constitutive law of the material is bilinear, and the material properties are: $E=28.3$ GPa, $\nu=1/3$, $G_{ik}=490$ J/m$^2$, $k_r=8$, $s_0=0.0254$ (linear elastic limit stretch). A plane stress condition has been assumed.

The macroscopic material data $E$ and $G_{ik}$ are used to define the bond constitutive parameters, $c$, $s_c$, $s_0$, as explained in reference [12]. The bonds connected to the nodes to which the displacement is imposed are simply linearly elastic and have no possibility to weaken. In this way we avoid ruptures of the material where the displacements are directly applied.

In Fig. 6 the deformation of the structure at the end of the load history is represented. It is apparent that the crack has grown considerably and it has almost cut in two the plate. In the same figure the displacements have been amplified by a factor of 4. Finally in Fig. 7 a load versus displacement curve is shown: it represents the values of the average displacements on the top nodes and of the computed total vertical reaction on the same nodes.

Fig. 8 shows the damage index distribution in the area of the plate close to the crack. In an ideal case with infinitely dense grid and with bonds in all direction of the plane, the nodes close to the crack would be characterized by an index value of 0.5, as previously explained.
In the present case of grid with a horizon δ equal to three times the grid spacing Δx, the damage index values, with fully developed crack, are below 0.5 because bonds parallel to the crack are not broken and they correspond to a finite fraction of the total number of bonds of a single node.

Example 2

A second example is presented: a tensile test has been simulated on a pre-cracked specimen with an inclined crack at its centre as shown in Fig. 9 where the geometry and boundary conditions are represented: the external load is an imposed displacement, uniformly distributed on the top and bottom sides of the plate.

The dimensions of the specimen are: W=49 mm, H=50 mm, t=2.5 mm (thickness) and the crack, which is in the middle of the plate, has a length of a=23 mm and it forms an angle α of 45° with the plate sides. The chosen constitutive law of the material is bilinear, and the material properties are: $E=28.3$ GPa, $\nu=1/3$, $G_c=490$ J/m², $k_r=8$, $s_0=0.0254$. Plane stress conditions have been assumed.

In Figures 10 and 11 the deformations of the plate immediately before and after the ultimate failure step is represented (with an amplification factor of 20 on the nodal displacements), whereas in Fig. 12 the total load (sum of all the nodal contributions on the top part) versus displacement diagram is shown. Finally Fig. 13 shows the damage index distribution in the plate, which is concentrated along the propagation path of the crack.
It is interesting to notice that damage index values are higher than in the previous case because, as shown in Fig. 11, a few points seem to be almost completely detached from the rest of the plate and therefore most of their bonds are broken.

Example 3

As last example, a double cantilever beam test has been simulated on a pre-cracked squared specimen with an inclined crack in its left side, as shown in Fig. 14.

The dimensions of the specimen are: \( W=50 \) mm, \( V=17.5 \) mm, \( t=2.5 \) mm (thickness) and the crack, which starts from the left side of the plate, has a length of \( a=15 \) mm and it forms an angle of 45° with the plate sides. The chosen constitutive law of the material is bilinear, and the material properties are: \( E=28.3 \) GPa, \( \nu =1/3 \), \( G_c =490 \) J/m², \( k_c =8 \), \( s_0 =0.0254 \). Again, plane stress conditions have been assumed.

In Fig. 15 the deformation of the plate at the end of the load history (with an amplification factor of 7 on the nodal displacements) is shown, whereas in Fig. 16 the external load versus displacement curve of the top left node is shown.
Finally Fig. 17 shows the damage index distribution on an area close to the initial crack, which is concentrated along the propagation path of the crack itself.

![Damage index distribution](image)

**Fig. 17: View of the damaged zone.**

4 Conclusions

The peridynamic theory is a powerful method to study the behaviour of solid bodies with a non-local and mesh free approach. The theory is based on integral equations so it is able to solve problems in the presence of discontinuities such as cracks, voids and inclusions. The peridynamic theory allows a clear definition of the concept of local damage as the ratio in each material point between the number of broken bonds and the initial number of bonds. The paper presents an ‘equilibrium implementation’ of the peridynamic theory to solve static problems with an implicit Newton Raphson approach. The proposed method allows, compared to the technique based on explicit dynamic relaxation, a reduced computational time. Moreover since an implicit method is used the convergence checks of the implicit method add an equilibrium control on the numerical solution, which is not usually available in explicit dynamic solutions.

The paper has presented the results obtained by applying the newly developed method to the study of non-linear problems with quasi-static crack propagation for three systems under imposed displacement conditions. Also in these cases the static solution predicts the crack path in a simple way. No particular algorithms have to be developed to deal with material discontinuities because the peridynamic approach is naturally suitable for discontinuous cases.

In the authors’ opinion peridynamics based computational methods are very promising in dealing with problems in which damage evolution has to be closely followed and monitored with no a priori knowledge on possible crack path propagation. Therefore they could be of the highest importance to be able to fully describe the damage process affecting structural components in order to evaluate their life expectancy for a safe but economically sound utilisation.

References


