1 Introduction

Functionally graded materials (FGM) are advanced composite materials formed of two or more constituents with a continuously variable composition which results a continuous variation of material properties from one surface of the material to the other. First introduced for high-temperature aerospace applications, the concept of FGM is currently actively explored in a variety of engineering areas, and methods and tools have been developed to analyse and design of structural elements incorporating FGMs, see reviews [1,2]. In coatings technology, the FGM concept is used to eliminate the mismatch between material properties at the coating/substrate interface, typical of conventional coatings, thus increasing coating’s resistance to functional failure in the form of cracking and debonding. Elastic flexural behaviour of coated plates with functionally graded coatings has been investigated both for rectangular [3-6] and circular plates [7], however only simple loading cases were considered in these studies. This paper presents a three-dimensional (3-D) elasticity analysis of coated plates with coating properties graded in thickness direction under three different types of loading: Uniformly Distributed Loading (UDL), Patch loading, and Point loading. A combination of analytical and computation means is used to study stress and displacements fields in the coated plates subjected to these transverse loadings.

2 Methodology

The analysis is based on the 3-D elasticity solution for a functionally graded rectangular coated plate developed by Kashtalyan and Menshykova [3-5] on the basis of the solution [8]. Here, this approach is extended to enable investigation of graded coatings under a variety of loading conditions and to study the combined effect of coating structure and loading type on the elastic deformation of coated plates.

A single layer coated plate of length $a$, width $b$, and total thickness $h$ is referred to the Cartesian coordinate system $Ox_1x_2x_3$. The substrate, of the thickness $h^{(1)}$ ($0 \leq x_1 \leq h^{(1)}$), and the coating, of the thickness $h^{(2)} = h - h^{(1)}$ ($h^{(1)} \leq x_1 \leq h$), are assumed to be perfectly bonded to each other and to have constant Poisson’s ratios $\nu^{(1)} = \nu^{(2)} = \text{const}$. The shear moduli of the substrate $(k = 1)$ and the coating $(k = 2)$ vary exponentially through the thickness according

$$G^{(k)}(x_2) = g^{(k)} \exp\left(\gamma^{(k)} \frac{x_2}{h}\right), \quad k = 1, 2$$

where $\gamma^{(k)}$ are the inhomogeneity parameters, and $g^{(k)} = G^{(k)}(h)$.

The loading, $Q(x_1, x_2)$, is applied to the upper surface of coating, while the bottom surface of the plate remains free. It is assumed that the loading can be expressed as a double Fourier series

$$Q(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{\pi nx_1}{a} \sin \frac{\pi mx_2}{b}$$

Various loading schemes can be investigated using this method. In particular, uniformly distributed loading $Q(x_1, x_2) = q_0$ yields

$$q_{mn} = \frac{16q_0}{\pi^2 mn}, \quad m,n = 1,3,5,...$$

For a point force $P$ applied at the point $(x_1^0, x_2^0, h)$ we have $Q(x_1, x_2) = \frac{P}{ab} \delta(x_1 - x_1^0) \delta(x_2 - x_2^0)$, with

$$q_{mn} = \frac{4P}{ab} \sin \frac{\pi mx_0}{a} \sin \frac{\pi nx_0}{b}, \quad m,n = 1,2,3,...$$

For patch loading $Q(x_1, x_2) = q_0$ applied at the centre of the plate $-c \leq x_1 \leq c, -d \leq x_2 \leq d$, we have
\[ q_{mn} = \frac{16q_0}{\pi^2 mn} \sin \frac{\pi mx_1}{a} \sin \frac{\pi ny_2}{b} \sin \frac{\pi mc}{a} \sin \frac{\pi md}{b} \]

\[ m,n = 1,3,5,\ldots \]  

Three-dimensional displacement field in the plate can be represented in terms of two displacement functions \([3-5]\) as

\[
u^{(k)} \Delta - \frac{\partial^2}{\partial x_1^2} \frac{\partial L^{(k)}}{\partial x_1} + \frac{\partial N^{(k)}}{\partial x_1} = \frac{1}{2G^{(k)}} \left( \nu^{(k)} \Delta - \frac{\partial^2}{\partial x_1^2} \frac{\partial L^{(k)}}{\partial x_1} - \frac{\partial N^{(k)}}{\partial x_1} \right),
\]

\[
u^{(k)} \Delta - \frac{\partial^2}{\partial x_2^2} \frac{\partial L^{(k)}}{\partial x_2} - \frac{\partial N^{(k)}}{\partial x_2} = \frac{1}{2G^{(k)}} \left( \nu^{(k)} \Delta - \frac{\partial^2}{\partial x_2^2} \frac{\partial L^{(k)}}{\partial x_2} \right)
\]

\[
u^{(k)} \Delta - \frac{\partial^2}{\partial x_3^2} \frac{\partial L^{(k)}}{\partial x_3} + \frac{\partial}{\partial x_3} \left[ \frac{1}{2G^{(k)}} \nu^{(k)} \Delta - \frac{\partial^2}{\partial x_3^2} \frac{\partial L^{(k)}}{\partial x_3} \right] = \frac{1}{2G^{(k)}} \left( \nu^{(k)} \Delta - \frac{\partial^2}{\partial x_3^2} \frac{\partial L^{(k)}}{\partial x_3} \right) \]  

From Eq. (6) and the constitutive equations for an isotropic inhomogeneous material, the components of the stress tensor can be expressed in terms of displacement functions \(L^{(k)}\) and \(N^{(k)}\) as

\[
\sigma_{33}^{(k)} = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) L^{(k)},
\]

\[
\sigma_{13}^{(k)} = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \frac{\partial L^{(k)}}{\partial x_3} + G^{(k)} \frac{\partial^2 N^{(k)}}{\partial x_3 \partial x_1},
\]

\[
\sigma_{23}^{(k)} = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \frac{\partial L^{(k)}}{\partial x_3 \partial x_2} - G^{(k)} \frac{\partial^2 N^{(k)}}{\partial x_3 \partial x_2},
\]

\[
\sigma_{11}^{(k)} = \left( \nu^{(k)} \Delta + \frac{\partial^4}{\partial x_1^2 \partial x_1} \right) L^{(k)} + 2G^{(k)} \frac{\partial^2 N^{(k)}}{\partial x_1^2},
\]

\[
\sigma_{22}^{(k)} = \left( \nu^{(k)} \Delta + \frac{\partial^4}{\partial x_2^2 \partial x_2} \right) L^{(k)} - 2G^{(k)} \frac{\partial^2 N^{(k)}}{\partial x_2^2},
\]

\[
\sigma_{12}^{(k)} = -\left( \nu^{(k)} \Delta - \frac{\partial^2}{\partial x_1^2} \right) \frac{\partial L^{(k)}}{\partial x_2} + \frac{\partial^2}{\partial x_1^2} \frac{\partial L^{(k)}}{\partial x_2} - G^{(k)} \frac{\partial^2 N^{(k)}}{\partial x_1^2},
\]

Functions \(L^{(k)}\) and \(N^{(k)}\) must satisfy the following partial differential equations

\[
\Delta \left( \frac{1}{G^{(k)}} \Delta L^{(k)} \right) - \frac{1}{1-\nu^{(k)}} \left( \Delta - \frac{\partial^2}{\partial x_1^2} \right) L^{(k)} \frac{d^2}{dx_1^2} \left( \frac{1}{G^{(k)}} \right) = 0,
\]

\[
\Delta N^{(k)} + \frac{d}{dx_3} \ln G^{(k)}(x_3) \frac{\partial N^{(k)}}{\partial x_3} = 0 \quad (8)
\]

If the edges of the plate are assumed to be simply supported, the variables in the functions \(L^{(k)}\) and \(N^{(k)}\) can be set as

\[
L^{(k)}(x_1,x_2,x_3) = \psi_1^{(k)}(x_1,x_2) \phi_1^{(k)}(x_3),
\]

\[
N^{(k)}(x_1,x_2,x_3) = \psi_2^{(k)}(x_1,x_2) \phi_2^{(k)}(x_3)
\]

This yield functions \(\psi_1^{(k)}\) and \(\psi_2^{(k)}\) in the form

\[
\psi_1^{(k)}(x_1,x_2) = \sin \frac{\pi mx_1}{a} \sin \frac{\pi nx_2}{b},
\]

\[
\psi_2^{(k)}(x_1,x_2) = \cos \frac{\pi mx_1}{a} \cos \frac{\pi nx_2}{b}
\]

Functions \(\phi_1^{(k)}\) and \(\phi_2^{(k)}\) are found from the following ordinary differential equations

\[
h^2 x^2 \frac{d^3 \phi_1^{(k)}}{dx_3^3} - 2\nu^{(k)} h^3 \frac{d^4 \phi_1^{(k)}}{dx_3^4} + 2\nu^{(k)} h^4 \frac{d^5 \phi_1^{(k)}}{dx_3^5} +
\]

\[
[\nu^{(k)} - 2\alpha^2 h^2] h^2 x^2 \frac{d^3 \phi_2^{(k)}}{dx_3^3} + 2\nu^{(k)} h^4 \frac{d^5 \phi_1^{(k)}}{dx_3^5} +
\]

\[
+ \alpha^2 h^2 \left( \alpha^2 h^2 + \nu^{(k)} \right) \frac{d^{(2)} \phi_2^{(k)}}{dx_3^4} \right) \frac{d^{(2)} \phi_2^{(k)}}{dx_3^4} = 0,
\]

\[
h^2 x^2 \frac{d^3 \phi_2^{(k)}}{dx_3^3} - \alpha^2 h^2 \frac{d^{(2)} \phi_2^{(k)}}{dx_3^4} = 0
\]

The displacement methods function presented above leads to the following representations for displacement and stress fields in a coated plate:

\[
u^{(k)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{j,m}^{(k)} U_{j,m}(x_3) \cos \frac{\pi mx_1}{a} \sin \frac{\pi nx_2}{b},
\]

\[
\sigma_{rt}^{(k)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{j,m}^{(k)} P_{j,m}(x_3) \sin \frac{\pi mx_1}{a} \cos \frac{\pi nx_2}{b},
\]

\[
r,t = 1,2,3
\]

Here, \(A_{j,m}^{(k)}\) are sets of 12 arbitrary constants, which for any pair of \(m\) and \(n\) can be determined from the stress and displacement continuity conditions at the coating/substrate interface and boundary conditions at the top surface of the coating and bottom surface of the substrate. These conditions lead to a set of 12 equations solved in MATLAB.

For displacements, functions \(U_{i,j,m}^{(k)}\) are
\[ U_{1,\text{mon}}(x_3) = - \frac{q_{\text{mon}} h}{2g^{(k)}} \frac{\pi n h}{a} \exp[-\gamma^{(k)}(x_3 - 1)] \times \left[ -\nu^{(k)} \alpha h^2 f^{(k)}(x_3) + (\nu^{(k)} - 1) \frac{d^2}{dx_3^2} f^{(k)}(x_3) \right], \]

\[ U_{2,\text{mon}}(x_3) = - \frac{q_{\text{mon}} h}{2g^{(k)}} \frac{\pi n h}{b} \exp[-\gamma^{(k)}(x_3 - 1)] \times \left[ -\nu^{(k)} \alpha h^2 f^{(k)}(x_3) + (\nu^{(k)} - 1) \frac{d^2}{dx_3^2} f^{(k)}(x_3) \right], \]

\[ U_{3,\text{mon}}(x_3) = - \frac{q_{\text{mon}} h}{2g^{(k)}} \frac{\pi n h}{a} \exp[-\gamma^{(k)}(x_3 - 1)] \times \left[ (\nu^{(k)} - 1) \left( -\nu^{(k)} \frac{d^2}{dx_3^2} f^{(k)}(x_3) + \frac{d}{dx_3} f^{(k)}(x_3) \right) \right. \]

\[ - \alpha^2 h^2 \left( (\nu^{(k)} - 2) \frac{d}{dx_3} f^{(k)}(x_3) - \nu^{(k)} \gamma^{(k)} f^{(k)}(x_3) \right), \]

\[ j = 1, \ldots, 4; \]

\[ U_{1,\text{mon}}(x_3) = - \frac{q_{\text{mon}} h}{g^{(k)}} \frac{\pi n h}{a} \frac{d}{dx_3} f^{(k)}(x_3), \]

\[ U_{2,\text{mon}}(x_3) = \frac{q_{\text{mon}} h}{g^{(k)}} \frac{\pi n h}{a} f^{(k)}(x_3), \quad U_{3,\text{mon}}(x_3) = 0, \quad j = 5, 6; \]

For stresses, functions \( P_{ij,\text{mon}}^{(k)} \) are

\[ P_{33,\text{mon}}^{(k)}(x_3) = q_{\text{mon}} \alpha^2 h^2 f^{(k)}(x_3), \]

\[ P_{13,\text{mon}}^{(k)}(x_3) = q_{\text{mon}} \alpha^2 h^2 \left( \frac{\pi n h}{a} \frac{d}{dx_3} f^{(k)}(x_3) \right), \]

\[ P_{23,\text{mon}}^{(k)}(x_3) = q_{\text{mon}} \alpha^2 h^2 \left( \frac{\pi n h}{a} \frac{d}{dx_3} f^{(k)}(x_3) \right), \]

\[ P_{11,\text{mon}}^{(k)}(x_3) = q_{\text{mon}} \nu^{(k)} \alpha^2 h^2 \left( \frac{\pi n h}{b} \right)^2 f^{(k)}(x_3), \]

\[ - \nu^{(k)} \left( \frac{\pi n h}{b} \right)^2 \frac{d^2}{dx_3^2} f^{(k)}(x_3) - \left( \frac{\pi n h}{a} \right)^2 \frac{d^2}{dx_3^2} f^{(k)}(x_3) \right], \]

\[ P_{22,\text{mon}}^{(k)}(x_3) = q_{\text{mon}} \nu^{(k)} \alpha^2 h^2 \left( \frac{\pi n h}{a} \right)^2 f^{(k)}(x_3), \]

\[ - \nu^{(k)} \left( \frac{\pi n h}{a} \right)^2 \frac{d^2}{dx_3^2} f^{(k)}(x_3) - \left( \frac{\pi n h}{b} \right)^2 \frac{d^2}{dx_3^2} f^{(k)}(x_3) \right]. \]

In the expressions above, \( x_3 = x / h \), and functions \( f^{(k)}(x_3) \) \( (j = 1, \ldots, 6) \) are

\[ f_1^{(k)}(x_3) = \exp \left( \frac{\gamma^{(k)} x_3}{2} \right) \cosh \frac{\mu^{(k)} x_3}{2}, \]

\[ f_2^{(k)}(x_3) = \exp \left( \frac{\gamma^{(k)} x_3}{2} \right) \sinh \frac{\mu^{(k)} x_3}{2}, \]

\[ f_3^{(k)}(x_3) = \exp \left( \frac{\gamma^{(k)} x_3}{2} \right) \cos \frac{\beta^{(k)} x_3}{2}, \]

\[ f_4^{(k)}(x_3) = \exp \left( \frac{\gamma^{(k)} x_3}{2} \right) \sin \frac{\beta^{(k)} x_3}{2}, \]

\[ f_5^{(k)}(x_3) = \exp \left( \frac{\gamma^{(k)} x_3}{2} \right) \cosh \frac{\beta^{(k)} x_3}{2}, \]

\[ f_6^{(k)}(x_3) = \exp \left( \frac{\gamma^{(k)} x_3}{2} \right) \sinh \frac{\beta^{(k)} x_3}{2}, \]

where
\[ \left( \lambda^{(k)} \right) = \frac{1}{2} \left( \pm \beta^{(k)} \right)^2 + \sqrt{\beta^{(k)^2} + \gamma^{(k)^2} \frac{\alpha^2 h^2 - V^{(k)}}{1 - V^{(k)}}}, \]

\[ \alpha = \pi \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2, \quad \beta^{(k)} = \frac{\gamma^{(k)^2}}{4} + \alpha^2 h^2. \]

3 Results and discussion

The extended solution, Eqs. (12), was validated through comparison with data available in the literature [3,9] as well as with predictions of a finite element model developed in ABAQUS which employs the user-implemented graded elements [10].

Table 1 shows comparison between the present model and Reddy’s plate theory [9] for a square \((a = b)\) simply supported plate with \(a/h = 10\) under UDL. The results are given for stresses and displacements normalised as follows:

\[ \bar{u}_3 = u_3(0.5a, 0.5b, h)(Eh^3 / a^2 q_0), \]
\[ \sigma_{11} = \sigma_{11}(0.5a, 0.5b, h)(h^2 / a^2 q_0), \]
\[ \sigma_{12} = \sigma_{12}(0, 0, h)(h^2 / a^2 q_0), \]
\[ \sigma_{13} = \sigma_{13}(0, 0.5b, h)(h / aq_0), \]

where

\[ E = 2G(1 + \nu) \]

Excellent agreement was found between analytical and finite element predictions.

On the basis of the extended solution presented in the previous section, a comparative study was carried out to establish the specifics of stress and displacements fields in coated plates under three types of loading: Uniformly Distributed loading, Patch loading and Point loading. To compute stresses and displacements, the infinite series in Eqs. (12) are truncated to finite sums, with the number of terms in the sum determined from a convergence study for a given type of loading. For UDL and Patch loading, the number of terms in \(m\) and \(n\) required to achieve convergence was found to be 29, for Point loading it was 49.

The results of the comparative study are presented in Figs. 1-6, where through-thickness variation of normalised stresses \(\bar{\sigma}_0 = \sigma_0 / q_0\) and normalised displacements \(\bar{u} = G_c u_3 / (q_0 h)\) is shown for plates with homogeneous coating (HC) and functionally graded coating (FGMC). The shear modulus ratio between the coating and substrate is taken as \(G_c / G_s = 10\), where \(G_c, G_s\) is the shear moduli of the homogeneous substrate and the homogeneous coating, respectively. In the graded coating, the shear modulus varies from the \(G_s\) value at the interface to the \(G_c\) value at the top surface. The homogeneous substrate is modelled as FGM with \(\gamma^{(i)} = 10^{-5}\) in Eq. (1).

For UDL and Patch loading, the length-to-thickness ratio of the plate was taken as \(a/h = b/h = 10\), and the patch size is taken as \(0.25a \times 0.25a\). For Point loading, the ratio \(a/h = b/h = 50\) was considered. The type of coating has a strong influence on the through thickness variation of the in-plane normal stress \(\bar{\sigma}_{11}\) (Fig. 1). As opposed to the HC case, where a large discontinuity in this stress component is present, in the FGMC case, variation of this stress component is continuous across the coating/substrate interface. However, while replacement of HC with FGMC eliminates the stress discontinuity, it leads to increase in stress magnitude at the top surface of the coating compared to the HC case. Similar dependency is observed for the in-plane shear stress \(\bar{\sigma}_{12}\) (Fig. 2).

The out-of-plane normal \(\bar{\sigma}_{33}\) and shear \(\bar{\sigma}_{13}\) stresses (Figs. 3, 4) are continuous across the coating/substrate interface due to perfect bonding between the coating and substrate. The type of transverse loading has little effect on the variation of \(\bar{\sigma}_{33}\) in both HC and FGMC cases. For the transverse shear stress \(\bar{\sigma}_{13}\), replacement of the homogeneous coating with a graded one reduces the magnitude of stress within the substrate in the vicinity of the interface, and also reduces the maximum value of this stress.

Through thickness variation of the in-plane displacement \(\bar{u}_3\) (Fig. 5) in both cases is almost linear for all considered types of transverse loading. The value of the normalized transverse displacement \(\bar{u}_3\) (Fig. 6) in the FGMC case is greater than in the HC case due to some loss of flexural stiffness associated with replacing a homogeneous coating with a graded one. Because of the large length-to-thickness ratio of the plate subjected to Point loading, transverse displacement is almost constant through the thickness in this case.
4 Acknowledgements

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5. References


Table 1. Comparison between the present model and Reddy’s plate theory [9] for a square \((a = b)\) simply supported plate with \(a/h = 10\) under UDL.

<table>
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<th>(u)</th>
<th>Present solution</th>
<th>Reddy’s theory [9]</th>
<th>Difference, %</th>
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<td>(\sigma_{11})</td>
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<td>0.2873</td>
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<td>0.15</td>
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<td>(\sigma_{13})</td>
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Fig. 1. Through-thickness variation of the normalised in-plane stress $\tilde{\sigma}_{11}(0.5a, 0.5b, x_3)$ in plates with homogeneous (HC) and functionally graded (FGMC) coatings.

Fig. 2. Through-thickness variation of the normalised in-plane shear stress $\tilde{\sigma}_{12}(0,0,x_3)$ in plates with homogeneous (HC) and functionally graded (FGMC) coatings.
Fig. 3. Through-thickness variation of the normalised out-of-plane normal stress $\bar{\sigma}_{33}(0.5a, 0.5b, x)$ in plates with homogeneous (HC) and functionally graded (FGMC) coatings.

Fig. 4. Through-thickness variation of the normalised transverse shear stress $\bar{\sigma}_{13}(0, 0.5b, x)$ in plates with homogeneous (HC) and functionally graded (FGMC) coatings.
Fig. 5. Through-thickness variation of the normalised in-plane displacement $\tilde{u}_1(0, 0.5b, x_i)$ in plates with homogeneous (HC) and functionally graded (FGMC) coatings.

Fig. 6. Through-thickness variation of the normalised transverse displacement $\tilde{u}_3(0.5a, 0.5b, x_i)$ in plates with homogeneous (HC) and functionally graded (FGMC) coatings.