INTRODUCTION

Use of composite materials is constantly increasing for last few decades. Aerospace, automobiles, oil and gas industry, civil construction, in fact all types of industries are opting for these stronger, stiffer and lighter materials. Although these materials possess many benefits but they also pose challenges which require extensive research to build knowledge and confidence in their use. Damage is one such area which is complex and requires in depth understanding. Matrix cracking is an important type of damage because it is usually the first form of damage and leads to complex problems for example increase in moisture absorption by material, delamination etc. This work is concerned with the reduction in mechanical properties of unidirectional composite materials in the presence of matrix micro cracks. Various approaches can be found in literature for the analysis of transversely cracked laminates. E.g. shear lag, variational approach, stress transfer, continuum damage mechanics, finite element and finite strip analysis. Most of them are generically restrictive in one respect or another. Some are simple but less accurate and others provide accuracy on the cost of complexity. Variational approach [1] has been very interesting but it is restricted to number of layers at a particular ply orientation e.g. cross ply. A recent work done [2] extends variational approach to angle ply laminates however it applies only to two plies which extends to third ply due to symmetry. For complex loading, more general layup and complicated scenario, analytical models are generally intractable. A semi analytical finite strip model [3] which can simulate scenarios closer to the practical situation or lay ups provide compromise between the crude but simple and accurate but complex approach. It involves finite strip discretisation in one of the axis i.e. (z-axis). It then further discretised x dimension using Fourier series. Boundary conditions were satisfied by variational principle. The second order Euler Lagrange differential equation is solved with the help of approximate solution based on algebraic equations. This current work employs finite strip discretisation in the z-direction exactly as done in [3]. However it solves the Euler Lagrange differential equation directly using Matlab inbuilt ODE solver. This requires traction boundary conditions to be prescribed which was not required in [3]. This improvement helps to avoid any discretization in x direction. It also simplifies the procedure by employing boundary conditions directly. The boundary conditions are clearly derived and formulated for cracked and uncracked surfaces. Then the Euler Lagrange equation is solved with the help of Matlab differential equation solver by applying boundary conditions appropriately. Displacement boundary conditions are applied as it is. Stress boundary conditions are transformed to be represented in form of displacements by using constitutive equations and strain displacement relationship. This makes the implementation of finite strip method simpler and provides opportunity to further generalize the finite strip method. It is already used and benefited from as shown in [4] where variational method for analysis of cracked laminate is extended.

STATEMENT OF THE PROBLEM

Displacement based finite strip formulation [3] is an effective but complex technique which involves Fourier approximation to solve the Euler Lagrange equation. This methodology can be further extended to analyze realistic and more complex layup with relaxed assumptions; however it requires simplification and clarity. The solution of Euler Lagrange equation can be obtained without approximation which is presented in this current work. Boundary conditions for the problem are clearly formulated. Then by developing computer code in Matlab the solution is obtained by solving Euler Lagrange equation which is 2nd Order ordinary differential equation. This makes finite strip solution clearer easier to understand and use. Boundary conditions are presented and discussed. Results are produced and compared with existing literature.
3. Formulation

Li et al [3] presented displacement based generalized plan strain approach which deals with cracked laminated composites having arbitrary layup. The strain energy per unit depth in y-axis in the laminate is presented as [3]. For details see [3]. Equations below are included for completeness.

\[
E = \frac{1}{2} \int \left( [\theta]^T \left[ k_0 \right] \theta + [\theta]^T \left[ k_1 \right] [\theta] + [\theta]^T \left[ k_2 \right] [\theta] - 2[\theta]^T \left( F_0 - 2[\theta]^T \left( F_1 \right) \right) \right) \, dx + \text{const.}
\]

(1)

Where for each finite strip the stiffness K and force F matrices are as follows

\[
\left[ k_0 \right] = \int \left[ b_0 \right]^T \left[ c \right] [l] \frac{dz}{d\zeta} \, d\zeta
\]

\[
\left[ k_1 \right] = \int \left[ b_0 \right]^T \left[ c \right] [l] \frac{dz}{d\zeta} \, d\zeta
\]

\[
\left[ k_2 \right] = \int \left[ b_0 \right]^T \left[ c \right] [l] \frac{dz}{d\zeta} \, d\zeta
\]

\[
[F_0] = -\int \left[ b_0 \right]^T \left[ c \right] [l] \frac{dz}{d\zeta} \, d\zeta
\]

\[
[F_1] = -\int \left[ b_1 \right]^T \left[ c \right] [l] \frac{dz}{d\zeta} \, d\zeta
\]

(2)

where B matrices are defined as

\[
\left[ b_0 \right] = \left[ b_{01}, b_{02}, b_{03} \right] \quad \left[ b_1 \right] = \left[ b_{11}, b_{12}, b_{13} \right]
\]

\[
\left[ b_{01} \right] = \left[ \begin{array}{c}
0 \\
0 \\
0 \\
N_1 \\
0 \\
0 \\
0 \\
\end{array} \right] \\
\left[ b_{02} \right] = \left[ \begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right] \\
\left[ b_{03} \right] = \left[ \begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right]
\]

(3)

\[
\left[ b_{11} \right] = \left[ \begin{array}{c}
0 \\
0 \\
0 \\
N_1 \\
0 \\
0 \\
0 \\
\end{array} \right] \\
\left[ b_{12} \right] = \left[ \begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right] \\
\left[ b_{13} \right] = \left[ \begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right]
\]

and displacement vectors are

\[
\left[ \theta \right] = \left[ \begin{array}{c}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right] \\
\left[ l \right] = \left[ \begin{array}{c}
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right]
\]

\[
\left[ \phi \right] = \left[ \begin{array}{c}
\theta \\
\phi \\
\phi \\
\phi \\
\phi \\
\phi \\
\phi \\
\end{array} \right]
\]

Strain vector is

\[
[\varepsilon] = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}]^T
\]

which can be written in the form of B matrices as

\[
[\varepsilon] = \left[ b_0 \right] [\varepsilon] + [b_1] [\phi] + [l] [\varepsilon]
\]

Where strains in terms of displacement and curvature are

\[
\varepsilon_x = \varepsilon_{x0} + 2\varepsilon_{x0} \frac{\partial U}{\partial x}
\]

\[
\varepsilon_y = \varepsilon_{y0} + 2\varepsilon_{y0} \frac{\partial W}{\partial y}
\]

\[
\varepsilon_z = \frac{\partial W}{\partial z}
\]

\[
\gamma_{yz} = \frac{\partial V}{\partial z}
\]

\[
\gamma_{xz} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}
\]

\[
\gamma_{xy} = \gamma_{xy0} + \varepsilon_{xy0} \frac{\partial V}{\partial x}
\]

(4)

(5)

(6)

(7)

(8)

and the Euler Lagrange equation for total potential energy eq 1 is

\[
-\left[ k_2 \right] [\phi] + \left[ k_1 \right] [\phi] + \left[ k_0 \right] [\phi] = [F_0]
\]

Which can be re written as first order differential equation

\[
\left[ \dot{\phi} \right] = \left[ \theta \right]
\]

\[
-\left[ k_2 \right] [\dot{\phi}] + \left[ k_1 \right] [\dot{\phi}] + [k_0] [\phi] - [F_0] = 0
\]

(9)

In [3] the author further discretises the model in one of the directions (x-axis) by using Fourier transformation which can be avoided if the above equation is solved directly. It requires formulation of appropriate boundary conditions and solving above equation directly by prescribing these boundary conditions. Next section
presents the stress and displacement conditions in form of traction boundary conditions.

4. Boundary Conditions

Consider a laminate with equally spaced cracks in the middle plies. It is assumed that the cracks exist from end to end in y-axis direction. In order to formulate and explain the boundary conditions, a unit cell is presented in Figure 1. The inset presents enlarged unit cell with two equally spaced cracks from a laminate. Due to symmetry only shaded part of the laminate (Figure 1) is required to be analyzed.

At $x=0$ i.e. along line C, due to reflection symmetry 

\[ U = 0 \]  

(10)

\[ V = 0 \]  

(11)

Due to rotational symmetry $c_\pi^2$ around symmetry line C [5]

\[ u_2 = -u_1; \]  

(12)

\[ v_2 = -v_1; \]  

(13)

Where $u$ and $v$ are displacements in $x$ direction and $y$ direction. Their subscript presents location of points along the line through the center of each crack.

For the uncracked part, $\sigma_{xz}=0$ due to translational and rotational symmetry as explained below.

From the continuity consideration as shown in the free body diagrams in Figure 2,

\[ \sigma_{xz}^{1h} = \sigma_{xz}^{2h} \quad \text{and} \quad \sigma_{xz}^{2a} = \sigma_{xz}^{2h}. \]

Periodic consideration as from the translational symmetry requires

\[ \sigma_{xz}^{1a} = \sigma_{xz}^{2a} \quad \text{and} \quad \sigma_{xz}^{1b} = \sigma_{xz}^{2b} \quad \text{hence} \]

\[ \sigma_{xz}^{1h} = \sigma_{xz}^{2h}. \]

The rotational symmetry about the central axis requires

\[ \sigma_{xz}^{1b} = -\sigma_{xz}^{2a}. \]

The only possibility is therefore

\[ \sigma_{xz}^{1b} = \sigma_{xz}^{2a} = 0 \]  

(13)

Hence by considering translational and rotational symmetry [5]

\[ \sigma_{xz}=0 \]  

(14)

Considering Figure 1 and traction continuity [6]

\[ \sigma_{1x}^{1} = \sigma_{1x}^{2} \]  

(15)

\[ \sigma_{1y}^{1} = \sigma_{1y}^{2} \]  

(16)

\[ \sigma_{2x}^{1} = \sigma_{2x}^{2} \]  

(17)

Displacement can be given as

\[ u_2 - u_1 = l\varepsilon_x \]  

(18)

\[ v_2 - v_1 = l\gamma_{xy} \]  

(19)

By using relationship (12) displacement boundary conditions at $x=\frac{l}{2}$ are

\[ u_2 = \frac{1}{2}l\varepsilon_x \]  

(20)

\[ v_2 = \frac{1}{2}l\gamma_{xy} \]  

(21)

For the cracked parts of the laminate, since cracked surface is traction free surface, one has

\[ \sigma_{xx}=0 \]

\[ \sigma_{xy}=0 \]

\[ \sigma_{xz}=0 \]  

(22)

Now by using above strain displacement equations 3,4,5,6,7 and 8 with constitutive equations below,

\[ \sigma_{ii} = C_{ij} \varepsilon_{ij} \]

boundary conditions on cracked and uncracked surface can be defined in terms of displacement
Hence displacement boundary conditions on all uncracked surfaces are

\[ U = 0 \]  
\[ V = 0 \]  
\[ C_{55} \frac{\partial W(x, z)}{\partial x} + C_{54} \frac{\partial V(x, z)}{\partial z} + C_{55} \frac{\partial U(x, z)}{\partial z} = 0 \]  

and displacement boundary conditions on the cracked surface are

\[ C_{61} \left( \epsilon_{x0} + 2z \kappa_6 + \frac{\partial U(x, z)}{\partial x} \right) + C_{62} \left( \epsilon_{y0} + 2z \kappa_6 + \frac{\partial U(x, z)}{\partial y} \right) + C_{63} \left( \frac{\partial W(x, z)}{\partial x} \right) = 0 \]  
\[ C_{64} \left( \epsilon_{x0} + 2z \kappa_6 + \frac{\partial V(x, z)}{\partial x} \right) + C_{65} \left( \epsilon_{y0} + 2z \kappa_6 + \frac{\partial V(x, z)}{\partial y} \right) + C_{66} \left( \frac{\partial U(x, z)}{\partial y} \right) = 0 \]  
\[ C_{55} \frac{\partial W(x, z)}{\partial x} + C_{54} \frac{\partial V(x, z)}{\partial z} + C_{55} \frac{\partial U(x, z)}{\partial z} = 0 \]  

In order to solve equation 9, \((6n+3)\) boundary conditions are required on each boundary. Each finite strip inherits 3 boundary conditions and total no of boundary conditions available are \(9n\).

For each of the nodal line on uncracked part, boundary conditions are equations 23,24,25 and at nodal line on crack surface these are 26,27 and 28. These boundary conditions are more in number as compared to requirement to solve differential equations, hence these are required to be applied appropriately with respect to three regions on the boundary, namely, Uncracked, cracked and crack tip as described below

**Uncracked boundary portion**

Boundary conditions are applied to each nodal line. However on last and first nodal lines of each finite strip average are taken of both strips sharing the nodal lines. First two boundary conditions are displacement type \(U\) and \(V\) (23)(24) which are both zeros and same in any two adjacent uncracked elements and plies due to continuity. However the third one is stress boundary condition presented in form of displacements (25). It is actually shear stress in the \(yz\) plane. Its value from two adjacent strips at same nodal line might not be the same. Adopting the practice in finite elements, it is recommended that average of two values from adjacent strips be used. Same is applicable to the neighbouring nodal lines on ply interface due to shear stress continuity.

**Cracked boundary portion**

On cracked part of boundary all three are stress boundary conditions(26)(27)(28), hence average values are used for neighbouring strips at the common nodal line. As cracks only span within plies of the same ply angle, there is no ply interface involved.

**Crack tips**

Crack tip is a unique and critical part of the unit cell. It is assumed to be part of uncracked laminate and boundary conditions derived for uncracked part of the laminate are applied on the nodal line at crack tip. In vicinity of the crack tip, neighbouring nodal lines fall in cracked and uncracked region. Crack tip is assumed as a part of uncracked laminate hence boundary conditions(23)(24)(25) for uncracked laminate are applied .Contrary to other two types of regions shear stress continuity is not assumed here and average of two values from adjacent strips is not applied.

**5. Results and Discussion**

Equation (9) is solved with the help of Matlab inbuilt ODE solver. Equations (23) to (28) are the boundary conditions needed to reach the solution. It is applied on various examples available in the literature and results are discussed in this section. Example 1 is a standard crossply situation with 90º plies in center with 0º plies on outside as shown in Figure 4. Due to assumed symmetry around z-axis, only half of the unit cell is presented in Figure 4. Material properties are presented in Table 1. In this analysis the loading is given by \(N_{xx}\) in normal direction. Results presented by Li et al [3] were based on approximate solution through Fourier series which are compared with those achieved by using boundary conditions formulated in this work and by employing Matlab® solver. The comparison is presented in Figure 5 to Figure 7.

Under uniaxial tension, the distributions of stresses \(\sigma_{xx}, \sigma_{zz}\) and \(\sigma_{xz}\) in the cracked plies at \(z=0, \frac{L}{2}\) and \(t\) are shown in Figure 5, Figure 6 and Figure 7 respectively. All stresses are normalized with respect to \(\sigma_{x0}\), the axial stress in 90º ply when the laminate is uncracked and subject to the same loading. These results are almost exactly same as compared to original work. They show
that the behavior of stresses is different at different z locations.

Above example presents analyses of graphite epoxy crossply under uniaxial tension. Same is analysed under pure shear loading by Hashin[1], however whole of the problem had to be analytically reformulated by the author. Current displacement based finite strip method can be applied directly since it has provision of applying same formulation for different loading situation including pure shear. Current scenario is same as first example apart from loading i.e pure shear.

Figure 8 to Figure 11 show the distribution of stresses $\sigma_{xy}$ and $\sigma_{xz}$ at the z positions indicated when the laminate is under pure shear. The stresses are normalized with respect to $\sigma_{xy0}$ developed in uncracked laminate subjected to the same loading.

In original work [3] each stress component is presented in same plot at all the z locations. In order to compare plots without mixing them up two plots for each of stress component are presented. Figure 8 and Figure 9 presents behaviour of $\sigma_{xy}$. Figure 10 and Figure 11 are for $\sigma_{xz}$. Again the results are very much similar to the original work [3] apart from the maximum values of these stresses on crack tips. In case of $\sigma_{xy}$ this solution provides maximum value of 3.5 which is around 4.5 in original work. Also $\sigma_{xy}$ is not predicted with very smooth curve at $z = t$ in original work. Again these approximate solutions show differences dependent on method employed e.g. Fourier series used in original work. The important conclusion is that the behaviour of stresses is very much complex on crack tip which is clearly presented by finite strip method and not predicted well by most of the methods presented in the literature.

Li et al. has compared only one example of non-crossply laminate which is presented first in [7]. Same example is used to present the application of the current formulation on non-crossply scenario. It is $[0°/-45°/+45°/90°]_s$ graphite/epoxy the data is included in table 1. It is assumed that material is under uniaxial loading in x-axis direction. The most important feature of this example is presentation of complex behaviour of the $\sigma_{xx}$ near crack tip. In [3], although this behaviour is observed and highlighted however stress does not vanish on crack tip. It was recommended to use more refined mesh and Fourier terms to achieve more realistic behaviour. In current scenario $\sigma_{xx}$ vanish on the crack surface and shows complex behaviour as well. It is achieved with same parameters used in the original work. It is the application of boundary conditions formulated in this work which helps to achieve it with Matlab® solver.

In order to get confidence in current mathematical model and computer program it is applied to other cases in literature. Data and results presented in [8] by using variational approach under bending load are also reproduced with the help of finite strip methodology. Material properties are presented in Table 2.

Figure 13 and Figure 14 shows plot of the $\sigma_{xx}, \sigma_{xz}$ and $\sigma_{xz}$ stresses for both glass epoxy and graphite/epoxy $[0°/90°]_s$. These stresses are plotted on $z = 0.5t$ which is mid-way through the thickness of the cracked 90° layer. Current analysis again shows good agreement for both laminates away from the crack tip. Apart from $\sigma_{xz}$, where variational results underestimates the stress values. Finite strip approach predicts rapid stress variation near crack tip and smooth and stable stress behaviour away from it which is more realistic. It is relevant to mention here that original work [8] is analytical and current formulation is refined to have four finite strips in each ply. Hence results achieved by current finite strip formulation cannot be exactly same as original work. Application of this finite strip methodology with improved boundary conditions shows that this method can be applied to different lay ups and loading conditions without the requirement of re formulation.

The stiffness reduction of the laminate with respect to the crack density is calculated for example 1 presented in. It is calculated by $E = \frac{N_{xx}}{T_{xx}}$, where $T$ is thickness of laminate and $E$ is effective Young’s modulus of the damage laminate. It is normalized with respect to $E_0$ i.e effective Young’s modulus of virgin laminate. It is compared with the Hashin’s prediction in Figure 15. Difference in two predictions is due to the fact that Hashin’s analysis provides lower bound and current analysis gives upper bound. Similarly prediction of shear stiffness reduction with respect to crack density is presented in Figure 16. It is obtained by defining effective shear modulus of the laminate as $G = \frac{N_{xz}}{T_{xz}}$. It is normalized with respect to $G_0$ which is effective shear modulus of the virgin laminate.

**6. Conclusions**

Finite strip based displacement based approach is effective semi analytical approach for evaluation of composite laminate with matrix crack. It is based on
generalized plain strain assumption which can be directly applied to any ply lap up. It has been previously accomplished by solving Euler Lagrange equation with the help of Fourier approximation [3]. It requires discretization of two of the axis with finite strip and Fourier transformation. Boundary conditions are not provided as Euler Lagrange equation is satisfied through variational method. It makes this methodology complex, less intuitive and restricts further development of this technique. This current work has solved same Euler Lagrange equation directly by prescribing boundary conditions. It does not need Fourier transformation. Boundary conditions are formulated in form of displacements and discussed fully on cracked and uncracked surface which has improved the methodology. This provides a stepping stone to extend finite strip approach to be further developed for even more practical composite systems. It opens up the option to formulate stress based approach to achieve lower bound for stiffness reduction in the presence of matrix cracks, which will be presented in future.

References
Table 1: Glass & graphite epoxy properties for modelling matrix cracking under bending[3]

<table>
<thead>
<tr>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphite/Epoxy</td>
<td>Graphite/Epoxy</td>
</tr>
<tr>
<td>$E_1$</td>
<td>208.3 GPa</td>
</tr>
<tr>
<td>$E_2$</td>
<td>6.5 GPa</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.255</td>
</tr>
<tr>
<td>$\nu_{23}$</td>
<td>0.413</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>1.65 GPa</td>
</tr>
<tr>
<td>$t$</td>
<td>0.203mm</td>
</tr>
<tr>
<td>$L$</td>
<td>0.406mm</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glass Epoxy</td>
<td>Graphite Epoxy</td>
</tr>
<tr>
<td>$E_1$</td>
<td>50.0 GPa</td>
</tr>
<tr>
<td>$E_2$</td>
<td>130.0 GPa</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.254</td>
</tr>
<tr>
<td>$\nu_{23}$</td>
<td>0.428</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>4.70 GPa</td>
</tr>
<tr>
<td>$t$</td>
<td>0.2 mm</td>
</tr>
<tr>
<td>$L$</td>
<td>2t</td>
</tr>
</tbody>
</table>

Table 2: Properties of different composite materials for analysis of matrix cracking[8]

<table>
<thead>
<tr>
<th>Glass Epoxy</th>
<th>Graphite Epoxy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$</td>
<td>50.0 GPa</td>
</tr>
<tr>
<td>$E_2$</td>
<td>15.20 GPa</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.254</td>
</tr>
<tr>
<td>$\nu_{23}$</td>
<td>0.428</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>4.70 GPa</td>
</tr>
<tr>
<td>$t$</td>
<td>0.2 mm</td>
</tr>
<tr>
<td>$L$</td>
<td>2t</td>
</tr>
</tbody>
</table>

Figure 1: Schematic presentation of a cracked Laminate for understanding of boundary conditions
Figure 2: Free body diagram of uncracked part to show stress continuity

Figure 3: Damaged laminate with equally spaced cracks
Figure 4: A graphite epoxy cross ply [0°/90°], with cracks in 90° plies.

Figure 5: Normalized axial stress distribution in x direction in 90° ply in cross ply formation

Figure 6: Normalized transverse direct stress distribution in x direction in 90° ply

Figure 7: Normalized transverse shear stress in x direction in 90° ply

Figure 8: Normalized in plane shear stress distribution in x direction in 90° ply
Figure 9: Normalized in plane shear stress distribution in x direction in 90º ply

Figure 10: Normalized transverse shear stress distribution in x-direction in 90º ply

Figure 11: Normalized transverse shear stress distribution in x direction in 90º ply

Figure 12: Normalized axial stress distribution in x direction at 90º ply
Figure 13: Stress distribution in Glass epoxy [0/90], at $z=0.5t$ under bending load

Figure 14: Stress distribution in Graphite epoxy [0/90], at $z=0.5t$ under bending load

Figure 15: Effective Young's modulus reduction of [0/90º], glass/epoxy laminate

Figure 16: Shear modulus reduction of [0/90º], glass/epoxy laminate