1 Introduction

Intense studies on damage problem of composite laminates revealed dependence of interlaminar fracture resistance on mixed-mode ratio and crack propagation direction with respect to fibers at the interface[1-3]. Andersons et al. summarized a lot of independent experimental studies on interlaminar fracture resistance under single mode I, II and III condition and concluded that the fracture resistance is highly dependent on angle between crack propagation direction and fiber orientation at the interface [1]. Volotin proposed to represent the dependence of fracture resistance using a second rank tensor, which can be easily transformed with orientation of the crack front [2]. Kondo et al. reported that fracture resistance depend upon crack growth direction due to formation of complex damage combining delamination with intra-laminar shear cracks under mixed mode II and III condition [3].

Many researchers are actively studying on numerical simulations of such a complex behavior of the material during the delamination growth. Cohesive element have been successfully used to simulate damage growth in those study[4-7]. Camanho et al. analyzed the problem of mixed-mode delamination growth under condition that the direction of crack propagation does not change [5]. Jiang et al. proposed concise constitutive law or interface element, which is based on an assumption that mode II and mode III components of fracture resistance are the same [6]. Kondo et al. demonstrated that the difference of 3 components of fracture resistance can be effectively dealt with a cohesive element in which a local coordinate system aligned to crack growth direction [7]. However, all of these methods do not consider out of plane rotation of the axis for mode decomposition, which may cause significant error in a case where out-of-plane deformation is large. One of important application cases with large out-of-plane deformation is delamination propagation in a buckled panel, for instance, a test specimen for CAI test.

In this paper, a formulation of cohesive element referring a local coordinate system whose axis are aligned to normal to the crack front and the interface after deformation. Then, some numerical examples are shown to demonstrate the feasibility of the formulation.

2 Formulation of Cohesive Element

2.1 Constitutive equation

The global and local coordinate systems are shown in Fig. 1. $x_i$ and $x'_i$ ($i=1, 2$) are global and local inplane coordinates respectively and $x_3$ is the coordinate normal to the interface. The unit base vectors are described as $e_i$ and $e'_i$ ($i=1, 2, 3$), respectively. The absolute relative displacement $w$ is derived from components of the relative displacement $w$ as

$$w^2 = w_1^2 + w_2^2 + w_3^2$$

Fig. 1 Global and local coordinate systems and process zone.
Since the crack shape after the propagation is not known before the analysis, the interlaminar fracture toughness must be determined according to the crack propagation direction and the current fiber direction. As the crack opening usually increases according to the distance from the crack front in the process zone as shown in Fig. 1, the crack front is thought to be normal to the direction of the maximum gradient of the relative displacement. Since the maximum gradient direction of \( w \) is same as that of \( w' \), \( w' \) is adopted instead of \( w \) for analytical convenience. The unit vector of the direction of maximum gradient is

\[
\mathbf{q} = \nabla w'.
\]  (2)

Inclination of normal to the crack surface cannot be neglected when out of plane deformation is large as shown in Fig. 2. To derive base vector \( \mathbf{e}'_i, 2 \) perpendicular vectors tangent to the crack surface \( s_i \) (\( i=1,2 \)) are introduced as follows.

\[
\mathbf{s}_i = \frac{\partial \mathbf{x}_1}{\partial \xi_i} \mathbf{e}_1 + \frac{\partial \mathbf{x}_2}{\partial \xi_i} \mathbf{e}_2 + \frac{\partial \mathbf{x}_3}{\partial \xi_i} \mathbf{e}_3
\]  (3)

where \( \xi_i (i=1, 2, 3) \) are coordinates at the midplane of top and bottom crack surface after deformation and \( \xi_i (i=1,2) \) are natural coordinates in the cohesive element. Vector \( \mathbf{n} \) normal to the surface is normal to both \( s_1 \) and \( s_2 \) as

\[
\mathbf{n} = s_1 \times s_2
\]  (4)

The base vector \( \mathbf{e}'_i \) (\( i=1,2,3 \)) are therefore determined as

\[
\mathbf{e}'_i = \frac{\mathbf{q}}{|\mathbf{q}|}
\]  (5)

\[
\mathbf{e}'_i = \frac{\mathbf{n}}{|\mathbf{n}|}
\]  (6)

\[
\mathbf{e}'_3 = \mathbf{e}'_1 \times \mathbf{e}'_2
\]  (7)

The relation between the local and global coordinate is,

\[
x' = Qx
\]  (8)

\[
Q_{ij} = \mathbf{e}'_j \cdot \mathbf{e}'_i = \frac{\partial x'_i}{\partial x_j}
\]  (9)

where \( x = [x_1, x_2, x_3]^T \) and \( x' = [x'_1, x'_2, x'_3]^T \).

Fig. 2 Global and local coordinate systems and process zone.

The condition of interfacial failure and the criterion for the crack growth are given by the following power forms with respect to the traction and energy release rate.

\[
\left( \frac{\tau'_1}{\tau'_{ic}} \right)^2 + \left( \frac{\tau'_2}{\tau'_{ic}} \right)^2 + U(w'_3) \left( \frac{\tau'_3}{\tau'_{ic}} \right)^2 = 1
\]  (10)

\[
U(w'_3) \left( \frac{G_{Ic}}{G_{Ic}} \right)^a + \left( \frac{G_{IIc}}{G_{IIc}} \right)^a + \left( \frac{G_{IIIc}}{G_{IIIc}} \right)^a = 1
\]  (11)

where \( \tau'_{ic} \) is the strength for the single load in \( x'_i \) direction and \( G_{Ic}, G_{IIc}, \) and \( G_{IIIc} \) are critical energy release rates in each mode. A unit step function \( U(w'_3) \) is used to realize the condition that compressive normal stress does not influence the interfacial failure.

As shown in Fig. 3, the fracture resistance being influenced by the fiber orientation according to the crack growth may be given as a function of angle \( \theta \) between crack growth direction and the material principal axis and angle \( \phi \) between the reinforcement directions of the plies forming this interface.

\[
G_i = G_i(\theta, \phi) \quad (i = I, II \text{ and } III)
\]  (12)

Fig. 3 Fracture resistance dependent on angle \( \theta \) and \( \phi \).
To consider the difference of the behavior of the interface in tension and compression, a modified absolute relative displacement and traction are introduced as

$$v^2 = w'^2 + w''^2 + w'''^2 \ U(w')$$

(13)

$$t^2 = t_1^2 + t_2^2 + t_3^2 \ U(w')$$

(14)

The relationships between the cohesive traction components \( t_i' \) 
\((i=1, 2, 3)\) and the relative displacement components \( w_i' \) 
\((i=1, 2, 3)\) in the local coordinate are given by a bilinear function in the cohesive element during loading as shown in Fig. 3. When the stiffness \( K \) is same in all direction before the softening starts, the modified absolute relative displacement \( v \) is also proportional to the absolute traction \( \tau \). When the ratio of each component \( w_i' \) to the modified absolute relative displacement \( v \) is \( m_i \), the critical absolute relative displacement \( w \) at which the softening starts is given by the following equation.

$$v^0 = \frac{1}{K} \left( \frac{m_1}{\tau_{1c}} \right)^2 + \left( \frac{m_2}{\tau_{2c}} \right)^2 + \left( \frac{m_3}{\tau_{3c}} \right)^2 + U(\bar{w})$$

(15)

Providing that the terminating displacement of softening path is simultaneously satisfied in all three direction, the terminating absolute relative displacement \( v_f \) is given as

$$v_f = \frac{2}{Kv^0} \left( \frac{m_1}{G_{Kc}} \ U(\bar{w}) + \left( \frac{m_2}{G_{Kc}} \right)^2 + \left( \frac{m_3}{G_{Kc}} \right)^2 \right)^{\frac{1}{2}}$$

(16)

The relationships between the cohesive traction components \( t_i' \) 
\((i=1, 2, 3)\) and the relative displacement components \( w_i' \) 
\((i=1, 2, 3)\) in the local coordinate are given by a bilinear function. As the initiation and the termination points of softening path are given and the ratio of each relative displacement component to the absolute relative displacement is maintained during this failure process of the cohesive element, we have the following constitutive relation.

$$\tau = \begin{cases} 
Kv \quad (v < v^0) \\
\frac{v_f - v}{v_f - v^0} \quad (v^0 < v < v_f) \\
0 \quad (v > v_f)
\end{cases}$$

(17)

where \( v^0 \) and \( v_f \) are relative displacements at the initiation and end of the failure, respectively.

When the ratio of each component \( w_i' \) to the modified absolute relative displacement \( v \) is \( m_i \), the traction components are given as

$$t_i' = m_i \tau \quad (i=1, 2, 3)$$

(18)

The following linear relationship between traction-relative displacement is given for the unloading and reloading.

$$\tau = \frac{v^0 (1 - d)}{v_{unloading}} \quad (v \leq v_{unloading})$$

(19)

where the damage parameter \( d \) is an non-decreasing value,

$$d = \max \left( \frac{v - v^0}{v_f - v^0} \right)$$

(20)

2.2 Finite element formulation

An eight-noded linear cohesive element was implemented as shown in Fig. 5. Components of displacement and nodal equivalent force at the nodes of the element are expressed as

$$\dot{u} = \left[ u_1 \quad u_2 \quad u_1 \quad u_2 \quad u_1 \quad u_2 \quad u_1 \quad u_2 \right]^T$$

(21)

$$\ddot{F} = \left[ F_1 \quad F_2 \quad F_3 \quad F_4 \quad F_5 \quad F_6 \quad F_7 \quad F_8 \right]^T$$

(22)
Upper and bottom indices of the variables denotes a node number and a reference coordinate system, respectively. Separation field in an element is defined as

$$\mathbf{w} = \mathbf{B}\hat{u}$$  \hspace{1cm} (23)

$$B^k_j = \delta^k_j \phi_k$$  \hspace{1cm} (24)

where \(\delta^k_j\) is cronecker’s delta. Lagrange polynomial function in terms of coordinate components \(\xi_1, \xi_2\) and \(\xi_3\) in element natural coordinate system are used as interpolation functions \(\phi_i\).

Jacobian matrix from the global coordinates to the natural coordinates of the element is defined as

$$J_{ij} = \frac{\partial x_i}{\partial \xi_j} = \frac{\partial B^k_i}{\partial \xi_j} u^k_j$$  \hspace{1cm} (25)

Components of the maximum gradient of the relative displacement \(\mathbf{q}\) in eq. 2 can be calculated with nodal displacements as

$$q_i = \frac{\partial}{\partial x_j} \left( w^2_1 + w^2_2 + w^2_3 \right)$$  \hspace{1cm} (26)

$$= 2w_j \frac{\partial w_j}{\partial x_i}$$  \hspace{1cm} (27)

$$= 2B^j_{jk} u^j_k \frac{\partial B^m_k}{\partial x_j} u^m_k$$  \hspace{1cm} (28)

$$= 2(\mathbf{B}\hat{u})^T \mathbf{B}_j \hat{u}$$  \hspace{1cm} (29)

where \(\mathbf{B}_j\) is partial derivative of the components of the \(\mathbf{B}\) matrix with respect to \(i\)th component of \(\mathbf{x}\). All components of this matrix have similar form like

$$\frac{\partial \phi_i}{\partial x_j} = \frac{\partial \phi_k}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_j}$$  \hspace{1cm} (30)

where \(\frac{\partial \xi_k}{\partial x_j}\) can be obtained from inverse of the jacobian matrix \(J\).

Now transformation matrix \(\mathbf{Q}\) can always be updated according to increment of the nodal displacements.

Following virtual work equations are derived from aforementioned relationship.

$$\delta\hat{u}^T \mathbf{F} = \int \delta \mathbf{w}^T \mathbf{t} d\mathbf{A}_e$$  \hspace{1cm} (31)

$$= \delta\hat{u}^T \int \mathbf{B}^T \mathbf{t} d\mathbf{A}_e$$  \hspace{1cm} (32)

where \(\delta\hat{u}\) is virtual nodal displacement, and \(\delta\mathbf{w}\) virtual relative displacement. Dot notation above \(\mathbf{F}\) and \(\mathbf{t}\) means differentiation in terms of time \(t\). Accordingly, equilibrium equations are derived as follows.

$$\dot{\mathbf{F}}_j = \int \dot{B}^j_{ik} \frac{\partial \tau_k}{\partial u^i_m} d\mathbf{A}_e u^i_m$$  \hspace{1cm} (33)

$$= K_{jk}^u u^i_m$$  \hspace{1cm} (34)

where \(K_{jk}^u\) is components of element tangent stiffness matrix. The stiffness matrix can be further developed as

$$K_{jk}^u = \int \int \dot{B}^j_{ik} \left( Q_{nm} \frac{\partial \tau_n}{\partial u^i_m} + \frac{\partial Q_{nm}}{\partial u^i_m} \tau_n \right) J d\xi_1 d\xi_2$$  \hspace{1cm} (35)

$$= \int \int \dot{B}^j_{ik} \left( \frac{\partial Q_{nm}}{\partial u^i_m} \tau_n \right) J d\xi_1 d\xi_2$$  \hspace{1cm} (36)

where \(J\) is Jacobian and

$$J = \left| \begin{array}{c} \frac{\partial \mathbf{x}}{\partial \xi_1} \\ \frac{\partial \mathbf{x}}{\partial \xi_2} \end{array} \right|. \hspace{1cm} (37)$$

Denoting the element tangent stiffness matrix as \(\mathbf{K}\), equilibrium equation can be expressed as

$$\dot{\mathbf{F}} = \mathbf{K}\hat{\mathbf{u}}$$  \hspace{1cm} (38)
\[ K = K_1 + K_2 \]  
where

\[ K_1 = \int_{\gamma_k} B^T Q^T DQ B \xi d \xi \]  
\[ K_2 = \int_{\gamma_k} B^T R \xi d \xi \]  

\[ D \text{ and } R \text{ indicate relation between traction and separation and effects of rotation of the local coordinate system on the tangent stiffness respectively. The components are} \]

\[ D_{ij} = \frac{\partial \tau_j^i}{\partial w_j} \]  
\[ R_i^j = \frac{\partial Q_j^i}{\partial u_j} \]  

3 Numerical analysis of delamination growth

3.1 Preliminary example

To confirm validity of the formulation of present element, especially regarding consideration of out-of-plane deformation, a simple model with large rotation shown in Fig. 6 was analyzed. The model contains 2 solid elements and a cohesive element between the solid elements. The rigid body movement of angle of 90 degree with respect to the axis of rotation combined with axial tension is applied by enforced displacement condition. Critical energy release rate for opening and shear modes are differently defined.

Fig. 6 A simple model with enforced rotation.

Fig. 7 shows relationship between magnitudes of load and separation of the cohesive element. The curve for the model with transformation became bi-linear curve as expected, but the model without transformation shows nonlinearity during the damage growth. Since normal to the crack surface rotates 90 degree, direction for mode decomposition is incorrectly recognized in the element without the transformation.

Although transformation matrix \( Q \) can be calculated in every iteration of Newton-Raphson scheme as described in the formulation in section 2.2, difficulty in converging the solution may arises with excessively frequent update of tangent matrix because it make the tangent too sensitive to displacement correction. In this example, the transformation matrix \( Q \) was kept constant during each increment and \( K_2 \) was omitted. In this way, the convergence was greatly improved with no difference in the result.

3.2 DCB test with consideration of dependence of fracture resistance on crack growth direction

A numerical example considering the change of crack front shape during crack propagation is shown. DCB test with fracture resistance dependent on crack growth direction was analyzed. Fig. 8 shows geometry and boundary conditions for the finite element model. Pre-crack is modeled with duplicated nodes at the interface and the cohesive elements are inserted into middle plane of the specimen. Material properties used in the model is shown in Table 2. \( G_k \) is represented as a function of angle between crack front and fibers at the interface as shown in Fig. 9 [3]. This function means that the
more the crack growth direction is inclined to fibers, the more the fracture resistance increases.

![FEA model for DCB test](image)

Fig. 8 FEA model for DCB test

Table 2 Material properties for the analysis

<table>
<thead>
<tr>
<th>$K$ (MPa)</th>
<th>$G_{Ic}$ (kJ/m$^2$)</th>
<th>$G_{IIc}$ (kJ/m$^2$)</th>
<th>$G_{IIIc}$ (kJ/m$^2$)</th>
<th>$G_{Ic0}$ (MPa)</th>
<th>$G_{IIc0}$ (MPa)</th>
<th>$G_{IIc90}$ (MPa)</th>
<th>$r_{IIc}$</th>
<th>$r_{IIIc}$</th>
<th>$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis 1</td>
<td>$10^7$</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>1</td>
</tr>
<tr>
<td>Analysis 2</td>
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<td></td>
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</tbody>
</table>

![Graph of $G_{Ic}(	heta)$](graph)

$G_{Ic}(	heta) = G_{Ic0} \cos^2 \theta + G_{Ic90} \sin^2 \theta$

$G_{Ic0} = 0.5$

$G_{Ic90} = 0.75$

Fig. 9 Dependence of fracture resistance on angle $\theta$

Comparison was made for cases with and without consideration of the dependence. Before that, energy release rate was calculated from the load-displacement curve from the analysis with area method for the case without the dependence to verify that the defined properties correctly reflected on the result. Area of the delamination was calculated with crack front shape interpolated with 4th-order polynomial function. The calculated value of energy release rate is 0.48 kJ/m$^2$, whose difference to the defined value is 4%. Difference of load-displacement curves in the cases with and without the dependence of fracture resistance is demonstrated in Fig. 10. By considering the dependence, load during the development of the crack increased 1.3% in average. Difference between crack front shapes for the 2 cases is also shown in Fig. 11. Analysis 2, in which the dependence of the fracture resistance was considered, shows 4% less amount of delaminated area than that in Analysis 1, in which the dependence is not considered. This is caused by the increase of fracture resistance due to change of crack front orientation.

![Comparison of load-displacement curve](graph)

Fig. 10 Comparison of load-displacement curve for the cases with and without the dependence

![Comparison of delamination growth](graph)

Fig. 11 Comparison of delamination growth for the cases with and without the dependence.
3.3 ENF test with consideration of the effect of large out-of-plane deformation

ENF test with large applied displacement was analyzed to validate the formulation of the present cohesive element considering geometric nonlinearity.

A finite element model and boundary condition used in the analysis are shown in Fig. 12. The material is assumed to be isotropic. 8-noded linear solid elements were used to model the specimen and delamination are modeled by inserting the cohesive elements into the middle plane of the specimen. No cohesive elements were inserted in the region of pre-crack and the nodes at the interface were kept duplicated. Automatic contact condition is defined at interface in the pre-cracked region to avoid interpenetration. Same values are used as material properties as the analysis of DCB test except that the dependence of the fracture resistance on growth direction is not considered.

Fig. 12 FEA model for ENF test.

Although the curves are not much different until displacement becomes about 35 mm, where the delamination propagates in the region with finer mesh, the difference is significant after the cohesive elements in region with coarse mesh started to fail because of the large interpenetration as shown in Fig. 14. This result implies importance of consideration of geometric nonlinear effect in problem of delamination propagation in a flexural media.
Fig. 15 Comparison of load-displacement curves for conventional and the present method.

4 Conclusion

Cohesive element in which fracture resistance is determined according to the crack growth direction is proposed. The local coordinate system is determined from gradient of relative displacement in the cohesive element so that its one axis is normal to the crack front and in the interface after deformation.

Numerical examples with the present cohesive element demonstrated that the dependence of fracture resistance on crack growth direction can be considered by introducing the present formulation. It is also demonstrated that unnatural deformation due to large rotation of the surface normal of the cohesive element is avoided with the present formulation.

The present cohesive element can be effectively used to analyze delamination growth in a laminate with 3 different modes of fracture resistance under large out-of-plane deformation.

References


