1 General Introduction

Tensile tests in multidirectional laminates are usually applied in cases where coupling effects do not exist or when they are negligible. The problem is related to the restraint caused by clamping devices: At one end the displacement is imposed in the direction of the axis of the testing machine. Nevertheless, the grips at both ends do not allow the free deformation of the specimen.

In the case of a unidirectional off-axis test, shear strains are prevented and consequently shear forces and bending moments appear. Figure 1 shows a unidirectional off-axis specimen loaded in pure tension and the same specimen assuming that both ends are clamped and only axial displacement is allowed at one end.

The problem of the unidirectional off-axis tensile test has been analysed by several authors, as it is used as a test method for obtaining in-plane shear properties of unidirectional composites [1,2].

In the case of multidirectional laminates with all types of coupling effects, membrane shear strains and bending and twisting curvatures appear. As the grips prevent these deformation modes, shear forces, bending moments and twisting moments must act at the ends of the specimen. Figure 2 shows a multidirectional specimen under tensile loading at one end and clamped at the other end, showing bending and twisting curvatures due to coupling effects.

The stress and displacement fields of a multidirectional laminate subjected to a tensile test has been recently obtained by applying Engesser-Castigliano theorems [3], after determining the complementary strain energy as a function of force and moment resultants of the laminate, including hygrothermal effects. The analysis has been carried out assuming that the gripping system and the tabs allow elastic rotations at the ends of the specimen. An experimental procedure for determining clamping compliances based on the measurement of strains and displacements at some points of the specimen has been proposed. The aim of this article is to show the main aspects of that analytic approach, to explain the procedure for obtaining the end compliances and to analyze by numerical examples the influence of the clamping conditions.

2 Tensile test in a multidirectional laminate

Composite laminates are cured in many cases at temperatures greater than the room temperature at laboratory. Considering the most general case of a multidirectional laminate, composites have normal and shear residual strains and bending and twisting curvatures. Figure 3 shows a carbon/epoxy laminate strip [80/-10] of two layers, where residual bending and twisting curvatures appear.
In a tensile test, when the specimen is placed in the grips, residual normal and shear strains are not prevented. Nevertheless, residual bending and twisting curvatures are constrained in order to adopt the plane geometry that corresponds to the gripping system. Then, when the specimen is placed at the grips, bending and twisting moments act at specimen ends in order to restrain residual curvatures. Therefore, the specimen is acted on by moments without applied tensile load. Nevertheless, normal and shear forces do not appear before applying load.

When the tensile load is applied, coupling effects tend to generate in-plane shear strains and bending and twisting curvatures. In this case, in-plane shear strains are restrained and then shear forces appear, besides twisting and flatwise bending moments. In-plane shear forces generate also edgewise bending moments in the plane of the laminate. Therefore, the application of load generates flatwise and edgewise bending moments, twisting moments and in-plane shear forces. In the analytical approach of this study, the ends are assumed to be linearly flexible with respect to rotations. Three kind of ends, depending on their flexibility will be treated through the work:

1. Clamped ends: The flexibility at the ends is null.
2. Flexible ends: The flexibility at the ends has a finite value.
3. Free ends: The flexibility at the ends is infinite. It is equivalent to the load applied in Figure 1(a).

### 3 Analytic approach

#### 3.1 Moments and forces in a cross section

Figure 4 shows the geometry of a tensile specimen with strip geometry clamped at both ends. The displacement is imposed at end A, applying a force $P$.

Figure 5 shows the forces and moments applied at the ends of the specimen. Lowercase letters indicate that they correspond to the whole cross section. The index notation corresponds to Daniel and Ishai [6]; $q$ subscript corresponds to $yz$, $r$ corresponds to $zx$ and $s$ corresponds to $xy$.

There are five redundant unknowns located at the end A of the specimen: In-plane and out-of plane shear forces $v_r$ and $n_s$, respectively, flatwise and edgewise bending moments $m_x$, respectively, and twisting moment $m_z$. By equilibrium in any cross section, it results that $v_r$, $n_s$ and $m_z$ are uniform and that $m_x$ and $m_y$ vary linearly along the length of the specimen

$$ n_i(x) = P, \quad m_i(x) = m_i $$  \hspace{1cm} (1)

It is assumed that rotations are partially prevented and that the displacement related to $n_s$ force is totally prevented. It is also assumed that compliances are the same at both ends. Then, the rotated angles at the ends are given by

$$ \theta_{ax} = C_i m_{ax}, \quad \theta_{ay} = C_i m_{ay} $$

$$ \theta_{az} = C_i m_{az} $$  \hspace{1cm} (2)

where $\theta_i$ are rotated angles and $C_i$ are their respective compliances.
3.2 Moments and forces per unit length

With respect to forces and moments per unit length, it is assumed that \( V_r, M_x, \) and \( M_s \) are uniform along the width of the specimen. According to equilibrium considerations, it results that \( V_q = M_y = 0 \). Due to bending moments, it is assumed that in any cross section normal forces per unit length \( N_s \) vary linearly along the width. For equilibrium considerations, the distribution of \( N_s \) is parabolic along the width. Resultant forces and moments per unit length in a cross section at a distance \( x \) from \( A \) are

\[
\begin{align*}
\begin{bmatrix} N_s \\ N_y \\ N_x \end{bmatrix} &= \begin{bmatrix} \frac{p}{b} + \frac{12}{b^2} (m_{s,x} + n_{x}) y \\ 0 \\ \frac{6n_{y}}{b} - \frac{b^2}{4} y \end{bmatrix} \\
\begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix} &= \begin{bmatrix} 1 \frac{b}{b^2} (m_{s,y} + v_{x,x}) \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} V_r \\ V_y \\ V_x \end{bmatrix} &= \begin{bmatrix} 0 \\ v_{y} \\ \frac{v_{x}}{b} \end{bmatrix}
\end{align*}
\]

3.3 Engesser-Castigliano theorems

Engesser’s first theorem or the generalized form of Castigliano’s second theorem states that:

\[
\frac{\partial U'}{\partial P_i} = U_{ri}^{*} = \delta_i
\]

Where \( P_i \) is a generalized load applied at point \( k \) and \( \delta_i \) is the generalized displacement at point \( k \) in the direction of \( P_i \). Generalized loads include load and moments and generalized displacements include displacements and rotations [4].

Otherwise, if \( X \) is a redundant force its associated displacement is null. Then, Engesser’s second theorem or the generalized form of Castigliano’s theorem of least work states that

\[
\frac{\partial U'}{\partial X} = U_{rX}^{*} = 0
\]

The derivative of the complementary strain energy of a multidirectional laminate has been determined in a recent work [5] in terms of resultant forces and moments of the laminate. The derivative corresponding to in-plane stresses is

\[
U_{rX}^{*} = \int_{A}^{b} \int_{A}^{L} \left\{ \begin{bmatrix} N_{s}^{T} \end{bmatrix} [K] \begin{bmatrix} N_{s} \\ V_{y} \\ M_{x} \end{bmatrix} + \begin{bmatrix} V_{y}^{T} \end{bmatrix} [M] \begin{bmatrix} N_{s} \\ V_{y} \\ M_{x} \end{bmatrix} \right\} dx dy
\]

Otherwise, the complementary strain energy that corresponds to the flexible ends is

\[
U_{rX}^{*} = \frac{1}{2} \left[ C \left( m_{s,i} + m_{s} \right) + C \left( m_{s,i} + m_{s} \right) + C \left( m_{s,i} + m_{s} \right) \right]
\]

Derivating Equation (7) with respect to \( P_k \) it results

\[
\frac{\partial U_{rX}^{*}}{\partial P_{k}} = C \left( m_{s,k} m_{s,k} + m_{s} m_{s} \right) + C \left( m_{s,k} m_{s,k} + m_{s} m_{s} \right) + C \left( m_{s,k} m_{s,k} + m_{s} m_{s} \right)
\]

With respect to out-of-plane stresses, taking into account Equations (3), the following conditions are satisfied

\[
\begin{align*}
\begin{bmatrix} M_{s,x} \\ M_{s,y} \\ M_{s,z} \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} M_{x} \\ M_{y} \\ M_{z} \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]

Then, the derivative of the complementary energy can be written as [5]

\[
U_{rX}^{*} = \int_{A}^{b} \int_{A}^{L} \left\{ \begin{bmatrix} V_{y}^{T} \end{bmatrix} [K] \begin{bmatrix} V_{y} \end{bmatrix} \right\} dx dy
\]

Where the components of the matrix \([K]\) are equivalent shear compliance coefficients.

3.4 Redundant unknowns

The redundant unknowns are obtained by applying Engesser’s second theorem. Out-of-plane shear forces \( v_{x} \) are null and in-plane shear force and edgewise bending moment are related by

\[
m_{s,x} = \frac{n_{y} L}{2}
\]

Resultant forces and moments per unit length are

\[
\begin{align*}
\begin{bmatrix} N_{s,x} \\ N_{s,y} \\ N_{s,z} \end{bmatrix} &= \begin{bmatrix} N_{s,x} + \frac{6}{b} N_{s,z} (2x - L) y \\ 0 \\ \frac{6N_{s,z}}{b^2} (2 - y^2) \end{bmatrix} \\
\begin{bmatrix} M_{x} \\ M_{y} \\ M_{z} \end{bmatrix} &= \begin{bmatrix} \frac{m_{s,x}}{b} \\ 0 \\ \frac{m_{s,z}}{b} \end{bmatrix}
\end{align*}
\]

Where \( N_{s,x} = \frac{F}{2} \) and \( N_{s,y} = \frac{F}{2} \). The unknowns \( n_{s,x}, m_{x} \) and \( m_{s,z} \) are obtained by applying Engesser’s first theorem. They are related by the following system of equations
\[ \left( c^2 a_x + \frac{6}{5} c_y + \frac{1}{2} c \right) N_{0x} + b_y M_x + b_x M_y = -a_x N_{0x} \]
\[ c_y N_{0x} + \left( d_z + 2 \frac{c}{c} d_a M_x + d_a M_y = -c_y N_{0x} - \kappa_{xt} \right) \]
\[ c_y N_{0x} + d_a M_x + d_a M_y + \left( d_y + 2 \frac{c}{c} d_a M_x + d_a M_y = -c_y N_{0x} - \kappa_{yt} \right) \]

Being \( c \) the length-to-width ratio; \( a_y \), \( b_y \) and \( c_y \) the compliance coefficients of the laminate; and \( \kappa_{xt} \) the hygrothermal curvatures.

### 3.5 Displacements and strains

The displacement in \( z \) direction \( \delta_z \) and the longitudinal strain \( \varepsilon_x \), obtained by differentiation of the displacement in \( x \) direction, are used as experimental measurements for the determination of end compliances.

The displacement field is obtained by applying Engesser’s first theorem with the unit load method. The displacement in \( x \) direction is

\[ u = (x - L) + a_x N_{0x} + \left[ \frac{1}{2} a_x + \frac{6}{5} c_y + 6 \frac{c}{c} a_x Y \right] + M_x (b_x + z d_x) + M_y (b_x + z d_x) + C M_y / C N_y b L y \]

The displacement field given in Equation (14) is valid for any value of \( N_{0x}, M_x \) and \( M_y \) applied at the end A. In the case that A is a flexible end, Equation (13) is satisfied and it results that

\[ u = L (x_0 - 1) + a_x N_{0x} + \left[ a_x + 3 c a_x N_{0x} b_y M_x + b_y M_y \right] + b_y (x_0 - 1) C M_x + L b_h y C M_{h0} \]

Where \( x_0 \), \( y_0 \) and \( z_0 \) are the following normalized coordinates

\[ x_0 = \frac{x}{L}, \quad y_0 = \frac{y}{h}, \quad z_0 = \frac{z}{h} \]

\[ c = \frac{L}{b} \] is the length-to-width ratio, \( h = \frac{b}{2}, L = \frac{L}{2} \).

The longitudinal strain \( \varepsilon_x \) can be obtained by derivation from Equation (15)

\[ \varepsilon_x = -a_x N_{0x} + \left[ a_x + 3 c a_x Y (2 x_0 - 1) \right] + b_y M_x + b_y M_x - \frac{b_y L}{z_0} C M_x \]

The displacement in \( z \) direction is

\[ w = -\frac{(L - \xi)}{2} \left[ \kappa_{xt} + c_y N_{0x} + c_y N_{0x} \right] + d_a M_x + d_a M_y \]

\[ + (L - \xi) \left[ \kappa_{yt} + c_y N_{0x} + c_y N_{0x} + d_a M_x + d_a M_y \right] \]

\[ - b (L - \xi) C M_x + \eta b C M_y \]

The displacement given in Equation (17) is valid for any value of \( N_{0x}, M_x \) and \( M_y \) applied at the end A. In the case that A is a flexible end, Equation (13) is satisfied and it results that

\[ w = L b y (x_0 - 1) C M_x + b y (2 x_0 - 1) C M_y \]

### 4 Determination of end compliances

#### 4.1 Motivation

Two limit cases can be considered with respect to end compliances: If the ends are clamped, compliances are null. Otherwise, if the ends are free with respect to the end rotations, the moments are null. Both limit cases can be compared in order to analyse end constraint effects. If the difference is not negligible, it is necessary to include the compliances \( C_z, C_x \) and \( C_y \) in the analysis. It is assumed that the end compliances depend on the stiffness of the tabs and the clamping system. Then, it is assumed that end compliances do not depend on the laminate configuration of the specimen.

An experimental procedure is proposed in order to determine those end compliance coefficients. In order to illustrate the methodology proposed, numerical values for AS4/3501-6 composite material have been considered. The calculation process is carried out by spreadsheets. In order to fix maximum load values for calculations, maximum tensile loads that correspond to a safety factor SF = 0.99 according to Tsai-Wu criterion are determined. Otherwise, moisture concentration is assumed to be null and the variation of temperature is assumed to be \( \Delta T = -150 \) °C. In all cases, laminates have 8 plies of 0.125 mm thickness. Tables 1 and 2 show material properties [6].

\[ \left( c^2 a_x + \frac{6}{5} c_y + \frac{1}{2} c \right) N_{0x} + b_y M_x + b_x M_y = -a_x N_{0x} \]
\[ c_y N_{0x} + \left( d_z + 2 \frac{c}{c} d_a M_x + d_a M_y = -c_y N_{0x} - \kappa_{xt} \right) \]
\[ c_y N_{0x} + d_a M_x + d_a M_y + \left( d_y + 2 \frac{c}{c} d_a M_x + d_a M_y = -c_y N_{0x} - \kappa_{yt} \right) \]
Table 1. Thermoelastic properties of AS4/3501-6.

<table>
<thead>
<tr>
<th>E₁ (GPa)</th>
<th>E₂ (GPa)</th>
<th>G₁₂ (GPa)</th>
<th>ν₁₂</th>
<th>ν₂₁</th>
<th>α₁ (°C⁻¹)</th>
<th>α₂ (°C⁻¹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>147</td>
<td>10.3</td>
<td>7</td>
<td>0.27</td>
<td>0.019</td>
<td>-9·10⁻⁶</td>
<td>2.7·10⁻⁵</td>
</tr>
</tbody>
</table>

Table 2. Strength properties of AS4/3501-6.

<table>
<thead>
<tr>
<th>F₁t (MPa)</th>
<th>F₁c (MPa)</th>
<th>F₂t (MPa)</th>
<th>F₂c (MPa)</th>
<th>F₆ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2280</td>
<td>1725</td>
<td>57</td>
<td>228</td>
<td>76</td>
</tr>
</tbody>
</table>

4.2 Unidirectional laminates and determination of Cz

In a unidirectional off-axis specimen, normal-shear coupling occurs due to the compliance coefficient aₓₓ. Otherwise, bₓ = cᵧ = Kₓᵧ = Kᵧₓ = 0. According to Equation (13)₂ and (13)₃ it results that mₓ = mᵧ = 0. From Equation (13)₁ the shear force nₓ is

\[ nₓ = -\frac{aₓₓ}{c^2aₓₓ + 1.2aₓₓ + 0.5CₓLb} P \]  \hspace{1cm} (19)

In the clamped case, when Cₓ = 0, the value of nₓ increases when c decreases. Otherwise, compliance matrixes are

\[ \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{h} [S] \quad [c] = [0] \quad [d] = \frac{12}{h^3} [S] \]

Being \([S] = [Q]^{-1}\) the compliance matrix of a unidirectional ply. Assuming clamped ends it results

\[ nₓ = -\frac{Sₓₓ}{c^2Sₓₓ + 1.2Sₓₓ} P \]  \hspace{1cm} (20)

The result of Equation (20) agrees with that obtained in Mujika [2]. In this case, the unique elastic rotation at the ends is related to mₓ. By two strain gages located at two points Z₁ and Z₂ of normalized coordinates Z₁(1,1,±1) and Z₂(0,1,±1) respectively, subtracting the corresponding strains obtained from Equation (16) it results

\[ \varepsilonₓ₁ - \varepsilonₓ₂ = \frac{6c}{b} aₓₓ nₓ \]  \hspace{1cm} (21)

From Equation (21) the value of nₓ is

\[ nₓ = \frac{b(\varepsilonₓ₁ - \varepsilonₓ₂)}{6caₓₓ} \]  \hspace{1cm} (22)

Equation (22) can be used in increment form, for two load vaules P₁ and P₂, being ΔP = P₂ - P₁. In this way, as residual strains do not depend on the applied load, their effect is eliminated. Then, Equation (22) can be written as

\[ Δnₓ = \frac{bΔεₓ}{6caₓₓ} \]  \hspace{1cm} (23)

where \(Δεₓ = [εₓ₁ - εₓ₂] - [εₓ₁ - εₓ₂]_c\)

Cₓ can be obtained after writing Equation (13)₁ in increment form, replacing \(Δnₓ\) given in Equation (23)

\[ Cₓ = \frac{2}{Lb} \left( \frac{aₓ}{Δnₓ} + \frac{c²aₓ}{aₓ + \frac{6}{5}aₓ} \right) \]  \hspace{1cm} (24)

A laminate with dimensions \(L = 200\) mm, \(b = 20\) mm is considered for simulating experimental measurements. As the maximum coupling coefficient aₓₓ occurs at \(θ = 35°\) this is the most appropriate angle for obtaining Cₓ. Otherwise, for the determination of in-plane shear properties \(θ = 10°\) is used.

For the clamped case, the coupling force nₓ is given by Equation (20). Replacing it in Equation (21) it results

\[ Δεₓ = -\frac{6}{b} aₓ Sₓₓ \Delta P \]  \hspace{1cm} (25)

The maximum strain difference occurs at the clamped case, as in the free case εₓ strains are uniform.

The maximum load is determined for SF = 0.99 in the clamped case at the point of coordinates \(x₀ = 0\) and \(y₀ = -1\), where the effect of edgewise bending moment is maximum.

Table 3 presents the compliance values obtained for \(θ = 35°\) with different degrees of clamping, ranging from 0% (free end) to 100% (clamped end) corresponding to \(ΔP = 1000\) N and. Being f the clamping factor, values of \(Δεₓ\) are obtained by multiplying by \(f\) the value \((1400·10⁻⁶)\) that corresponds to the clamped case obtained from Equation (25). The clamping factor is used in order to obtain intermediate values of Cₓ and it is not an absolute reference. Depending on specimen dimensions, strain that corresponds to the clamped case changes and end compliances corresponding to different values of \(f\).
Table 3. Compliance coefficients in a [35] laminate.

<table>
<thead>
<tr>
<th>Clamping factor, f(%)</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \varepsilon_x^z \times 10^{-6}$</td>
<td>0</td>
<td>280</td>
<td>560</td>
<td>840</td>
<td>1120</td>
<td>1400</td>
</tr>
<tr>
<td>$C_z (10^{-6}(N\cdot mm)^{-1})$</td>
<td>$\infty$</td>
<td>9.127</td>
<td>3.423</td>
<td>1.521</td>
<td>0.571</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4. Effect of the clamping factor in [10]a.

<table>
<thead>
<tr>
<th>Clamping factor, f(%)</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \varepsilon_x^z \times 10^{-6}$</td>
<td>0</td>
<td>109</td>
<td>219</td>
<td>328</td>
<td>438</td>
<td>547</td>
</tr>
<tr>
<td>$\Delta P$ (N)</td>
<td>0</td>
<td>3.455</td>
<td>6.909</td>
<td>10.36</td>
<td>13.82</td>
<td>17.27</td>
</tr>
<tr>
<td>$C_z (10^{-6}(N\cdot mm)^{-1})$</td>
<td>$\infty$</td>
<td>2.444</td>
<td>0.917</td>
<td>0.407</td>
<td>0.153</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4 presents several results obtained for $\theta = 10^\circ$ with different degrees of clamping, corresponding to $\Delta P = 1000$ N. The values of $\Delta \varepsilon_x^z$ are obtained as in the previous case and are lesser than in Table 3. Compliance coefficients for the same clamping degree are also lesser due to the lower value of $a_{xx}$.

4.3 Antisymmetric cross-ply laminates and determination of $C_x$

In an antysimmetric cross-ply laminate [0/90] tensile-bending coupling occurs due to $b_{xx}$ compliance coefficient.

Otherwise, $a_{xx} = b_{xx} = c_{xx} = c_{yy} = d_{xx} = d_{yy} = \kappa_{\text{eff}} = 0$. Therefore, according to Equations (13) and (13), it results that $a_y = m_y = 0$. From Equation (13), $M_y$ is

$$ M_y = -C_{xx} N_{xx} + \kappa_{\text{eff}} \frac{d_y}{d_{xx}} + 2C_{xx} \frac{\kappa_{\text{eff}}}{e} $$

When the specimen is placed in the grips, assuming $C_x = 0$, the residual moment is

$$ M_y^{\text{res}} = \frac{\kappa_{\text{eff}}}{d_{xx}} $$

The moment due to the load $P$ is:

$$ M_y = -\frac{
abla C_{xx}}{d_{xx}} N_{xx} $$

Moments in Equations (27) and (28) do not depend on the dimensions $L$ and $b$ of the specimen. Thus, they do not depend on the length-to-width ratio $c$. Otherwise they have opposite sign. Then, the residual moment compensates in some extent the moment generated by the load.

By two strain gages located at points $X_1$ and $X_2$ of coordinates $X_1(\frac{1}{2}, 0, -1)$ and $X_2(\frac{1}{2}, 0, 1)$, respectively, subtracting the corresponding strains obtained from Equation (16), it results

$$ \varepsilon_x^{\text{res}} - \varepsilon_x^{\text{ref}} = 2 \frac{b}{L} C_x m_y $$

From equation (29)

$$ C_x m_y = \left( \varepsilon_x^{\text{res}} - \varepsilon_x^{\text{ref}} \right) \frac{L}{2b} $$

$m_y$ can be determined from Equation (13) and replacing the value obtained in Equation (29).

$$ m_y = -\frac{1}{d_{xx}} \left( \Delta \varepsilon_x - \Delta \varepsilon_x^{\text{ref}} \right) L + c_y P + b \varepsilon_{xy}^{\text{ref}} $$

Applying Equation (31) at two different values of the load $P_1$ and $P_2$ and subtracting them it results

$$ \Delta m_y = -\frac{1}{d_{xx}} \left( \Delta \varepsilon_x L + c_y \Delta P \right) $$

Where

$$ \Delta \varepsilon_x = \left[ \varepsilon_x^{\text{res}} - \varepsilon_x^{\text{ref}} \right]_1 - \left[ \varepsilon_x^{\text{res}} - \varepsilon_x^{\text{ref}} \right]_2 $$

$C_x$ is obtained from the incremental form of Equation (30) being

$$ C_x = \frac{\Delta \varepsilon_x L}{2b \Delta m_y} $$

The compliance $C_x$ can be also determined by a measurement of the out-of-plane displacement $\delta_z$, according to Equation (18), in the point $X$ of coordinates $(\frac{1}{2}, 0, \pm 1)$

$$ C_x m_y = -\frac{4 \delta_z}{L} $$

From Equation (13), after replacing the value obtained in Equation (35)
\[ m_s = \frac{1}{2\pi} \left(\frac{8\delta^b}{E} - \frac{c_s P}{b\kappa}\right) \]  

In a similar manner than in the case of the measurement by strains, Equations (35) and (36) can be used in increment form in order to eliminate residual effects. The incremental form of Equation (36) is

\[ \Delta m_s = \frac{1}{2\pi} \left(\frac{8\Delta\delta^b}{E} - \frac{c_s \Delta P}{b\kappa}\right) \]

where

\[ \Delta\delta^b = [\delta^b]_e - [\delta^b]_i \]

The method of strain measurements has been used in order to simulate experiments. In the clamped case \( \Delta\varepsilon^x = 0 \) and in the free case the difference is maximum. \( \varepsilon_x \) in the free case can be obtained derivating Equation (14) and imposing that \( \varepsilon_{x0} = \varepsilon_{x0} = \varepsilon_{y0} = \varepsilon_{y0} = \kappa' = 0 \). Therefore, according to Equations (13) and (13) it results that that \( \varepsilon_{x0} = \varepsilon_{x0} = 0 \). From Equation (13) \( M_s \) is

\[ M_s = -\frac{c_s N_{ix} + \kappa''}{d_s + 2c_s} \]  

When the specimen is placed in the grips, assuming \( C_s = 0 \), the residual moment is

\[ M_s'' = -\kappa'' \]

The moment due to the load \( P \) is

\[ M_s' = -\frac{c_s N_{ix}}{d_s} \]

Moments in Equations (41) and (42) do not depend on the dimensions \( L \) and \( b \) of the specimen. Thus, they do not depend on the length-to-width ratio \( c \). Otherwise, they have opposite sign. Then, the residual moment compensates, in some extent, the moment generated by the load.

By the measurement of the \( \delta_e \) displacement at the points \( S_i(0,1,\pm1) \) and \( S_i(1,1,\pm1) \), according to Equation (18)

\[ C_s m_s = \frac{\delta^b_{S_i} - \delta^b_{S_i}}{b} \]

Table 5 presents the compliance values obtained for different degrees of clamping, ranging from 0% (free end) to 100% (clamped end). Being \( f \) the clamping factor, values of \( \Delta\varepsilon^x \) are obtained multiplying the value that correspond to the free case (-428·10^{-6}) by (1-f).

In this case, failure is related to \( \sigma_T \) stresses. The maximum \( \sigma_T \) of the free case is 6% greater with respect to the value of the clamped case. Otherwise, the longitudinal strain \( \varepsilon_x \) in the middle plane in the free case is 3% greater with respect to the value of the clamped case. Therefore, the consideration of flexible clamping in this case is not critical.

### 4.4 Antisymmetric angle-ply laminates and determination of \( C_s \)

In an antysymmetric angle-ply laminate \([\theta/\theta]_t\] tensile-twisting coupling occurs due to \( c_{xx} \) compliance coefficient. Otherwise, \( a_{xx} = b_{xx} = c_{xx} = c_{xx} = d_{xx} = \kappa'' = 0 \). Therefore, according to Equations (13) and (13) it results that that \( n_x = m_s = 0 \). From Equation (13) \( M_s \) is

\[ M_s = -\frac{c_s N_{ix} + \kappa''}{d_s + 2c_s} \]  

Moments in Equations (41) and (42) do not depend on the dimensions \( L \) and \( b \) of the specimen. Thus, they do not depend on the length-to-width ratio \( c \). Otherwise, they have opposite sign. Then, the residual moment compensates, in some extent, the moment generated by the load.

By the measurement of the \( \delta_e \) displacement at the points \( S_i(0,1,\pm1) \) and \( S_i(1,1,\pm1) \), according to Equation (18)

\[ C_s m_s = \frac{\delta^b_{S_i} - \delta^b_{S_i}}{b} \]  

From Equation (13), after replacing the value obtained in Equation (43)
Equations (43) and (44) can be used in increment form, subtracting equations that correspond to two load values \( P_1 \) and \( P_2 \), in order to eliminate the effect of residual displacements. Then the incremental form of Equation (44) is

\[
\Delta m_i = \frac{1}{d_{i}} \left( \frac{2(\delta^s_\Delta - \delta^0_\Delta)}{L} + c_{is} P + b \kappa_s \right) \tag{45}
\]

where

\[
\Delta \delta^s_\Delta = \left[ \delta^s_\Delta - \delta^0_\Delta \right]_L - \left[ \delta^s_\Delta - \delta^0_\Delta \right]_0 \tag{46}
\]

\( C_s \) is obtained from the incremental form of Equation (43) being

\[
C_s = \frac{\Delta \delta^s_\Delta L}{b \Delta m_i} \tag{47}
\]

In the clamped case \( \Delta \delta^s_\Delta = 0 \) and in the free case the difference is maximum. \( \delta^s_\Delta \) in the free case is obtained from Equation (17), imposing that \( n_s = m_s = m_s = 0 \). Then, the difference given in Equation (46) in the free case is

\[
\Delta \delta^s_\Delta = \frac{L}{2} c_{is} \Delta P \tag{48}
\]

Specimens of the type \([(\theta_1-\theta_2)]_{as} \) are considered. Dimensions are \( b = 20 \) mm and \( L = 200 \) mm. As the maximum of \( c_{xx} \) corresponds to \( \theta = 15^\circ \), this orientation is selected in order to maximize coupling effects. In the free case the maximum load that corresponds to \( SF = 0.99 \) is \( P_{max} = 15673 \) N. In the clamped case it is \( P_{max} = 23130 \) N. Then, the end constraint effects increase the strength of the laminate, as in the case of the cross-ply laminate. In the clamped case, end moments given in Equations (41) and (42) are

\[
M^s_{int} = 0.98 \text{ N·mm} \quad M^s = -16.3 \text{ N·mm} \]

Assuming that ends are free, the difference between displacements with \( \Delta P = 2000 \) N is \( \Delta \delta^s_\Delta = -2.342 \) mm. In the clamped case, \( \Delta \delta^s_\Delta = 0 \). Table 6 presents several results obtained for different degrees of clamping, ranging from 0% (free end) to 100% (clamped end). Being \( f \) the clamping factor, values of \( \Delta \delta^s_\Delta \) are obtained multiplying the value that correspond to the free case (-2.342 mm) by \( (1-f) \).

**Table 6. Effect of flexible ends in \([(15/-15)]_{as} \)**

<table>
<thead>
<tr>
<th>Clamping factor, ( f(%) )</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \delta^s_\Delta (\text{mm}) )</td>
<td>-2.342</td>
<td>-1.873</td>
<td>-1.405</td>
<td>-0.937</td>
<td>-0.468</td>
<td>0</td>
</tr>
<tr>
<td>( \Delta m (\text{N·mm}) )</td>
<td>0</td>
<td>-5.63</td>
<td>-11.3</td>
<td>-16.9</td>
<td>-22.5</td>
<td>-28.2</td>
</tr>
<tr>
<td>( C_s (10^6 \text{N·mm}^2) )</td>
<td>( \infty )</td>
<td>33.24</td>
<td>12.47</td>
<td>5.341</td>
<td>2.078</td>
<td>0</td>
</tr>
</tbody>
</table>

The failure is due to \( \tau_{LT} \) stresses at top and bottom lamina. The stress in the free case is 45% greater with respect to the value of the clamped case. Nevertheless, \( \varepsilon_t \) of the middle plane in the free case is only 2% greater than in the clamped case.

**5 Example: \[20/50]_4**

A laminate \[20/50]_4 of the same material and dimensions as in the previous cases has been analysed in order to see the influence of the clamping ends. Three different boundary conditions are analysed: Free ends, flexible ends and clamped ends. In the case of flexible ends, end compliance coefficients correspond to 60% of clamping, according to Tables 3, 5 and 6. Then:

\[
C_s = 1.521 \cdot 10^{-6} \quad C_s = 525.2 \cdot 10^{-6} \quad C_s = 5.541 \cdot 10^{-3}
\]

Two points of the specimen are considered for calculations: The central point \( A(x_0 = 0.5, y_0 = 0) \) and \( B(x_0 = 0, y_0 = -1) \) of maximum \( N_t \). The applied load corresponds to a \( SF = 0.99 \) at point A in the clamped case, being \( P = 2285 \) N. As coupling effects increase when the number of layers decreases \([7]\), this case can be considered critical. Table 7 presents end forces and moments per unit length and safety factors. The flexibility of the ends reduces the values of the end forces and moments.

**Table 7. End forces and moments in \[20/50]_4 \)**

<table>
<thead>
<tr>
<th>Clamping</th>
<th>( N_t (\text{N/mm}) )</th>
<th>( M_x (\text{N}) )</th>
<th>( M_y (\text{N}) )</th>
<th>SF(A)</th>
<th>SF(B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.73</td>
<td>0.73</td>
</tr>
<tr>
<td>Flexible</td>
<td>0.59</td>
<td>-9.23</td>
<td>-2.10</td>
<td>0.98</td>
<td>0.80</td>
</tr>
<tr>
<td>Clamped</td>
<td>1.033</td>
<td>-14.81</td>
<td>-5.56</td>
<td>0.99</td>
<td>0.72</td>
</tr>
</tbody>
</table>

At point A the maximum SF is for the clamped case and at point B the maximum SF is for the flexible case. Moreover, SF at point B is very similar for the flexible and clamped cases.

Figures 6-8 show stresses in the principal directions of orthotropy, \( \sigma_x, \sigma_y \) and \( \tau_{LT} \) at the critical point B. The values of SF at point B in Table 7 correspond to the bottom part of ply 8 in both, the free case (0.73)
and in the clamped case (0.72). $\sigma_T$ is greater in the clamped case (76.5 MPa) than in the free case (66.8 MPa). Otherwise, $\tau_{LT}$ is lesser in the clamped case (29.6 MPa) than in the free case (49.1 MPa). Then, the similarity in SF is not due to similar stress values, but for the combined effect of $\sigma_T$ and $\tau_{LT}$.

Figure 6. $\sigma_L$ stress in a [204/504] laminate at point B.

Figure 15 shows $\varepsilon_x$ strains along the thickness of the specimen at the central point A. In the case of the clamped case, the strain distribution is uniform. Strains are minimum in the clamped case and the relative differences of mean strains with respect to the clamped case are 40% in the free case and 15% in the flexible case.

Figure 7. $\sigma_T$ stress in a [204/504] laminate at point B.
The effect of the specimen dimensions has been discussed in the particular cases of unidirectional, antisymmetric cross-ply and angle-ply laminates. In the present example, for a given value of $N_{0x}$, if the length $L$ of the specimen decreases $N_{0s}$ increases and the change in $M_x$ and $M_s$ is very small. Increasing the width $b$ for the same value of $N_{0x}$, the effect is similar: $N_{0x}$ increases and the change in $M_s$ and $M_t$ is not appreciable. Figure 10 shows end force and moments for an applied load $N_{0x} = 100$ N/mm. Therefore, for a given laminate configuration, only $N_{0s}$ depends on the geometry of the specimen, increasing when $c$ decreases.
6 Conclusions

The compliances relatives to edgewise and flatwise bending and to twisting at the ends can be determined by strain and displacement measurements in unidirectional off-axis, antisymmetric cross-ply and angle-ply laminates, respectively.

Considering unidirectional laminates, the end shear force depends on the length-to-width ratio $c$. In the case of cross-ply and angle-ply laminates, end bending and twisting moments, respectively, do not depend on the length and the width of the specimen. Otherwise, residual effects compensate in some extent the end effects produced by the application of the load.

With respect to multidirectional laminates with general coupling, only the end shear force depends on the length-to-width ratio, as in the particular cases explained previously. Bending and twisting moments depend only on the laminate configuration. Otherwise, mean strains at the centre of the specimen depend on the flexibility of the ends.

7 References


