The high strength and stiffness of carbon nanotubes have generated enormous interest in the scientific community in recent years [1-3]. One of the areas has been the applicability of carbon nanotubes as a reinforcing constituent [4-14]. In fact, the unique atomic structure, high aspect ratio, light weight, extraordinary mechanical properties [15], and thermal conductivity make SWNT a potentially very attractive material to be used in polymer/carbon nanotube composite application. Carbon nanotubes (CNTs) have radii on the order of nanometers and lengths on the order of micrometers resulting in large aspect ratios beneficial to their use in composites. As the mechanical properties of composites depend directly upon the embedded fiber mechanical behavior, replacing conventional carbon fibers with CNTs can potentially improve composite properties, such as tensile strength and elastic modulus and can be a very promising candidate as the ideal reinforcing fibers for advanced composites.

Composites of carbon nanotubes dispersed in metallic or polymeric matrixes have attracted a considerable attention recently [16, 17]. Most studies on carbon nanotube reinforced composites (CNTRCs) have focused on their material properties [18-23]. Several investigations have shown that the addition of a small percentage of nanotubes (2~5% by weight) in a matrix may considerably increase the composite’s mechanical, electrical and thermal properties [19-23]. The effects of CNT dispersion and orientation [24], deformation mechanisms [25, 26], interfacial bonding [27, 28] on mechanical properties of CNT reinforced composites have been investigated experimentally. Wuire and Adali found that the stiffness of CNTRC beams can be improved significantly by the homogeneous dispersion of a small percentage of CNTs [29]. Vodenitcharova and Zhang studied the pure bending and bending-induced local buckling of CNTRC beams [30]. Formica et al. [31] presented the vibration behavior of CNTRC plates by employing an equivalent continuum model based on the Mori–Tanaka approach. They found that the improvement achieves a maximum when the carbon nanotubes are uniformly aligned with the loading direction. Instabilities in CNTRCs have also been of substantial interest and many experiments have observed buckling [32-34].

An increase in the utilization of nanocomposites is due to their physical properties which can be improved simultaneously. Physical properties such as low density, low thermal expansion, high resistance interfering with thermal shock and high strength in high temperatures attract researcher’s attention. Evidently, such composites are of paramount interest in aeronautic and astronautic technology, automobile and many other modern industries. High speed air craft structures are subjected not only to aerodynamic loading, but also to aerodynamic heating. The temperature rise may buckle the plate and exhaust the load caring capacity; therefore, elastic buckling of the composite plate reinforced CNTs corresponding to the practical application in air craft and car industries is too prominent.

Motivated by these considerations, the present work focuses attention on the thermal elastic buckling of composite plates reinforced
with carbon nanotubes subjected to in-plane uniform temperature. Due to this aim, we derive the governing equations of this plate and find a relation for critical load obtained by use of the principle of minimum potential energy, and then we study the effects of carbon nanotube volume fraction, radius of nanotubes, plate aspect ratio and the orientation angle of nanotubes in polymer matrix composite.

2 Equations
Total potential energy of the plate, due to the internal strain and the surface traction can be expressed as

$$V = \int_{R} U_{total} dR - \int_{T} T_{\mu} d\xi$$

(1)

Where \( U_{total} \) is defined as composite strain energy, \( R \) is volume of elastic body, \( T_{\mu} \) is \( \mu \)th component of surface traction, \( u_{i} \) is \( i \)th component of displacement, and \( St \) is a portion of plate surface on which surface traction is exerted.

The composite strain energy \( U_{total} \) is composed of strain energy in the matrix and CNTs, and the cohesive energy in CNT/matrix interfaces can be written as in the following [37]

$$U_{total} = U_{cohesive} + U = \frac{1}{2} \iint [\sigma] [\varepsilon] dV + \int_{\partial S} \phi dA$$

(2)

Where \( U_{cohesive} \) is the cohesive energy in the surface of the nanotube and polymer matrix and \( U \) is the strain energy in the matrix and nanotubes.

Considering the first stage of the cohesive law proposed by Tan et al. [37], the cohesive energy for CNT/matrix interface is obtained as below

$$U_{cohesive} = \int_{R} K_{f} J^{2} \varepsilon^{2} dV$$

(3)

Where \( R, f, \varepsilon \) are respectively radius of nanotube, carbon nanotube volume fraction, and equivalent stress of the composite plate [37]. \( K_{f} \) is the linear modulus of the interface [38]. \( J \) is a constant coefficient can be written as in the following

$$J = \left\{ \frac{0.3k_{c} \times 10^{9}}{3} + \frac{0.203}{R} - 0.25k_{c}f \times 10^{9} \right\}^{-1}$$

(4)

In Eq. (2), \( U \) in Cartesian coordinate system is defined as follow

$$U = \frac{1}{2} \sigma_{xx} \varepsilon_{xx} + \frac{1}{2} \sigma_{yy} \varepsilon_{yy} + \frac{1}{2} \sigma_{zz} \varepsilon_{zz} + \frac{1}{2} \tau_{xy} \varepsilon_{xy} + \frac{1}{2} \tau_{xz} \varepsilon_{xz} + \frac{1}{2} \tau_{yz} \varepsilon_{yz}$$

(5)

According to the principle of minimum potential energy, it necessary that stresses replace with strains and then strains transform into displacements by use of strain displacement relations. Therefore the Eq. (2) is rewritten in matrix form and takes the following form

$$U_{total} = \frac{1}{2} \iint [\varepsilon]^{T} [C] [\varepsilon] dV + \int_{R} K_{f} J^{2} \varepsilon^{2} dV$$

(6)

The constitutive relating the stress and the strain can be written as

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \\ C_{44} & C_{45} & C_{46} \\ C_{55} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

(7)

Where \( C_{ij} \) are transformed elastic coefficients in the global coordinate system.

Based on the classical plate theory and eliminating the shear deformation the constitutive relating the stress and the strain is as follow

$$\begin{bmatrix} \sigma_{x} \\ \sigma_{y} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{x} \\ \varepsilon_{y} \\ \gamma_{xy} \end{bmatrix}$$

(8)

The components of transformed elastic matrix are given as follow

$$\begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{21} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} = \begin{bmatrix} C_{11} \cos^{2} \theta + 2C_{12} \sin \theta \cos \theta + C_{16} \sin^{2} \theta \\ C_{21} \cos \theta \sin \theta + C_{22} \sin \theta \cos \theta + C_{26} \sin^{2} \theta \\ C_{16} \sin \theta + C_{26} \sin \theta \cos \theta + C_{66} \sin^{2} \theta \end{bmatrix}$$

(9)

Where \( \theta \) is the angle between orientation of fibers and the axial of the composite plate. The composite plate is assumed to be orthotropic.
The other parameters are calculated as follows:

\[ C_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}}, \quad C_{22} = \frac{E_2}{1 - \nu_{21} \nu_{12}}, \quad C_{66} = G_{12} \]

In the above formulas, \( E_1, E_2, \) and \( G_{12} \) are the longitudinal, transverse and shear moduli of the composite with the uniaxial straight fibers. \( \nu_{12} \) is the longitudinal transverse Poisson’s ratio. To simplify the analysis we will calculate these parameters using the rule of mixture [35].

\[ E_1 = E_m (1 - f) + E_f f, \quad E_2 = \left( \frac{1 - f}{E_m} + \frac{f}{E_f} \right)^{-1} \]

\[ G_{12} = \frac{G_m G_f}{G_m + G_f (1 - f)}, \quad G_m = \frac{E_m}{2(1 + \nu_m)}, \quad G_f = \frac{E_f}{2(1 + \nu_f)} \]

Where \( E_m, E_f, G_m, G_f \) and \( \nu_m, \nu_f \) are Young’s moduli and Poisson’s ratios of the matrix and the fibers. \( f \) is the volume fraction of the fibers.

The relations between the strain and the displacement in the classical plate theory are defined as

\[ \varepsilon_x = \frac{\partial u_0}{\partial x} + \frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = \frac{\partial v_0}{\partial y} + \frac{\partial^2 w}{\partial y^2}, \quad \gamma_{xy} = \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} - 2z \frac{\partial^2 w}{\partial x \partial y} \]

Where \( u_0 \) and \( v_0 \) are the displacements of the mid-plane in the \( x \) - and \( y \) - directions, respectively, which are assumed to be zero because there is no coupling between the in-plane and the out of plane displacements. \( w(x, y) \) denotes the displacement in the \( z \) - direction, i.e. the lateral deflection of the composite plate. Substituting Eq. (12) into Eq. (6) yields:

\[ U_{\text{total}} = \frac{1}{2} \int_0^a \int_0^b \left\{ \frac{z}{2} C_{11} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{z}{2} C_{12} \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} dxdy \]

Where \( a, b, \) and \( h \) are the length, the width, and the thickness of the rectangular composite plate respectively.

The work done by the surface traction is as follows:

\[ \int_{x_0}^{x_1} \int_{y_0}^{y_1} N_x \left[ \frac{\partial u_0}{\partial x} + \frac{1}{2} \frac{\partial^2 w}{\partial x^2} \right] dxdy + N_y \left[ \frac{\partial u_0}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} \right] dxdy \]

Where \( N_x \) and \( N_y \) are the resultant forces in the \( x \) - and \( y \) - directions, respectively and \( N_{xy} \) denotes the shear force. The composite plate is subjected to in-plane uniform temperature; therefore, \( N_{xy} \) is assumed to be zero. \( N_x \) and \( N_y \) are calculated as follows

\[ N_x = (C_{11} \alpha_x + C_{12} \alpha_y) h \Delta T \]

\[ N_y = (C_{22} \alpha_x + C_{21} \alpha_y) h \Delta T \]

\( \alpha_x, \alpha_y \) are respectively longitudinal and transverse thermal coefficients, respectively. Substituting Eqs. (13) and (14) in Eq. (1) yields the following simplified expression for the total potential energy subjected to in-plane uniform temperature.
\[ V = \frac{1}{2} \int_a^b \left( D_{11} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_{12} \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) + 2D_{22} \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + D_{66} \left( \frac{\partial^2 w}{\partial \alpha^2} \right)^2 + 2D_{66} \left( \frac{\partial^2 w}{\partial \alpha \partial y} \right)^2 + 2D_{26} \left( \frac{\partial^2 w}{\partial x \partial \alpha} \right)^2 \left( \frac{2\partial^2 w}{\partial \alpha^2} \right) + 2D_{26} \left( \frac{\partial^2 w}{\partial x \partial \alpha} \right)^2 \left( \frac{2\partial^2 w}{\partial \alpha \partial y} \right) + \frac{k_f J^2}{54R} \left[ \left( \frac{\partial^2 w}{\partial \alpha^2} \right)^2 + \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dx \right] (16) \\
- \int_a^b \int_b^0 \left[ (\overline{C}_1 \alpha_x + \overline{C}_2 \alpha_y) \Delta Th \left( \frac{\partial w}{\partial x} \right)^2 \right] dxdy \\
- \int_a^b \int_b^0 \left[ (\overline{C}_2 \alpha_x + \overline{C}_1 \alpha_y) \Delta Th \left( \frac{\partial w}{\partial y} \right)^2 \right] dxdy \\
\]

\[ D_{ij} \] is the bending stiffness matrix whose elements are defined as 
\[ D_{ij} = \int_0^\frac{z}{2} C_{ij} z^2 dz \] (17)

In this work, the plate is orthotropic, thus, \( D_{26}, D_{16} = 0 \).

2.1 Solution methodology

According to the levy solution [36], it is important to find suitable function for the lateral deflection. It is assumed that the lateral deflection can be written as the following separate function of \( x \) and \( y \) variables:
\[ w(x, y) = f(x)g(y) \] (18)

The following boundary conditions are assumed for all edged simply supported plate situation
\[ w = 0 \Rightarrow \begin{cases} x = 0 & 0 \leq y \leq b \\ x = a & 0 \leq y \leq b \\ 0 \leq x \leq a & y = 0 \\ 0 \leq x \leq a & y = b \end{cases} \] (19)

The solution function for buckling response is proposed as
\[ w(x, y) = \sum_{n=1}^\infty \sum_{m=1}^\infty A_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \] (20)

The assumed displacement field Eqs. (20) completely satisfies the boundary conditions specified in Eqs. (19). The substitution of the assumed solution function Eqs. (20) and its derivatives into Eqs. (16) provides a relationship as in the following
\[ V = \left( D_{11} + \frac{K_f J^2}{54R} \right) \left( \frac{2n\pi a}{b} \right)^2 \left( \frac{ab}{4} \right) \sum_{m=1}^\infty A_{nm}^2 + \left( D_{22} + \frac{K_f J^2}{54R} \right) \left( \frac{mnt b}{a} \right)^2 \left( \frac{ab}{4} \right) \sum_{n=1}^\infty A_{nm}^2 + \left( D_{11} + \frac{K_f J^2}{54R} \right) \left( \frac{mnt a}{b} \right)^2 \left( \frac{ab}{4} \right) \sum_{n=1}^\infty A_{nm}^2 \] (21)

For the plate to be in equilibrium, the total energy should be stationary
\[ \frac{\partial V}{\partial A_{nm}} = 0 \] (22)

Hence, solving Eq. (22) yield the following buckling load
\[ \Delta L_c = \left( \frac{K_f J^2}{54R} \right) \left( \frac{mnt a}{b} \right)^2 + D_{11} \left( \frac{mnt a}{b} \right)^2 + D_{22} \left( \frac{mnt b}{a} \right)^2 \] (23)

Another case considered in this work is an asymmetric (clamped supported at one edge and simple support on remaining edges) boundary condition as in the following
\[ w = 0 \Rightarrow \begin{cases} x = 0 & 0 \leq y \leq b \\ 0 \leq x \leq a & y = 0 \\ 0 \leq x \leq a & y = b \end{cases} \] (24)

The assumed displacement field which completely satisfies the boundary conditions specified in Eqs. (15) is proposed as in the following
The substitution of Eqs. (25) and its derivatives into Eqs. (16) and use of the principle of minimum potential energy lead to a relationship for critical load in the asymmetric boundary condition case as below

$$\Delta T_s = \frac{\alpha_s}{\alpha_c^2} \left[ \frac{(C_1^1 \alpha + C_1^2 \alpha)_h}{b^2} + \frac{(C_2^1 \alpha + C_1^2 \alpha)_h}{b^2} \right]$$

(26)

3 Result and discussion

In this work, the thermal buckling of a composite plate reinforced by CNTs subjected to in plane uniform temperature is investigated by the analytical method. Based on a relation for critical thermal buckling load obtained by use of the principle of minimum potential energy, the effects of carbon nanotubes volume fraction $f$, radius of nanotubes $R$, plate aspect ratio $a/b$ and the orientation angle of nanotubes $\theta$ in polymer matrix composite on the thermal buckling behavior of plates made of carbon nanotube-reinforced polymer composite materials are studied for two different kinds of boundary conditions. The composite plate is composed of polyethylene as the matrix with the Young’s modulus and the Poisson’s ratio of $E_m = 0.98 Gpa$, and $\nu_m = 0.33$, respectively. The CNTs are modeled as long, transversely isotropic fibers. The material properties of SWCNTs are $E_f = 1350 Gpa$ and $\nu_f = 0.26$.

Fig. 1 shows the effect of the CNTs volume fraction on the thermal buckling behavior of plates reinforced with SWCNTs for armchair carbon nanotubes (6, 6), the orientation angle of nanotubes $\theta = 45^\circ$, and two symmetric and asymmetric boundary conditions. It can be seen that the plate with volume fraction $f = 10\%$ has the highest critical thermal buckling load in comparison with volume ratio of $f = 1\%, 5\%$.

For constant aspect ratio and CNTs volume fraction, the highest critical thermal buckling load occurs for asymmetric boundary condition.

Fig. 2 shows the effect of CNT size on the critical load–aspect ratio curve. The orientation angle of nanotubes and the volume fraction is $\theta = 45^\circ$ and 10%, respectively. Three different armchair CNTs, (18, 18), (12, 12) and (6, 6), are studied, and the corresponding radii are $R = 1.25$, 0.83 and 0.42 nm, respectively. Small CNTs clearly give stronger reinforcing effect than large CNTs because, at a fixed CNT volume fraction, there are more small CNTs than large ones, and therefore there exist more interfaces. This observation of strong reinforcing effect for small CNTs also holds after CNTs are deboned from the matrix.

According to this fact, armchair carbon nanotubes (6, 6) are the best choice in comparison with armchair carbon nanotubes (18, 18) and (12, 12).

Fig. 3 shows the effect of the orientation angle of nanotubes on the critical load–aspect ratio curve. The volume fraction of armchair (6, 6) carbon nanotubes is 10%. From presented results, there is an optimum value for orientation angle of nanotubes. This value is 46° for symmetric boundary conditions and 42° for asymmetric boundary conditions.

In Fig. 4 the critical buckling load is plotted with respect to aspect ratio for different number of half- waves in the X direction in the case that the orientation angle of nanotubes and the volume fraction is $\theta = 45^\circ$ and 10%, respectively, for armchair carbon nanotubes (6, 6).

There is a minimum in Fig.4. By increasing the aspect ratio of the composite plate, the critical thermal buckling load decreases at the beginning and attains a minimum value and then it increases. The rate of critical load’s variations before reaching the minimum value is higher.
than the next step at which the critical load increases.

4 Conclusions
In this study, the thermal buckling of a composite plate reinforced by CNTs subjected to in plane uniform temperature is investigated by the analytical method. Based on a relation for critical thermal buckling load obtained by use of the principle of minimum potential energy, the effects of carbon nanotubes volume fraction, radius of nanotubes, the orientation angle of nanotubes, and the plate aspect ratio in polymer matrix composite are presented. Two different symmetric (simple supported at all edges) and asymmetric (clamped supported at one edge and simple support on remaining edges) boundary conditions have been considered.

The results show that the plate with volume fraction $f = 10\%$ has the highest critical thermal buckling load in comparison with volume ratio of $f = 1\%, 5\%$. For constant aspect ratio and CNTs volume fraction, the highest critical thermal buckling load occurs for asymmetric boundary condition. Armchair carbon nanotubes (6, 6) are the best choice in comparison with armchair carbon nanotubes (18, 18) and (12, 12). Small CNTs clearly give stronger reinforcing effect than large CNTs.

From presented results, there is an optimum value for orientation angle of nanotubes. This value is $46^\circ$ for symmetric boundary conditions and $42^\circ$ for asymmetric boundary conditions. By increasing the aspect ratio of the composite plate, the critical thermal buckling load decreases at the beginning and attains a minimum value and then it increases. The rate of critical load's variations before reaching the minimum value is higher than the next step at which the critical load increases.

![Fig. 1. The critical load– aspect ratio relation of a carbon nanotube reinforced polyethylene matrix composite. The volume fraction of armchair (6, 6) carbon nanotubes is 10%, 5% and 1%. The orientation angle of nanotubes is $\theta = 45^\circ$. (a) Symmetric boundary condition and (b) asymmetric boundary condition](image)
Fig. 2. The critical load–aspect ratio relation of a carbon nanotube reinforced polyethylene matrix composite plate. The orientation angle of nanotubes and the volume fraction is $\theta = 45^\circ$ and 10%, respectively, for three different armchair carbon nanotubes (18, 18), (12, 12) and (6, 6). (a) Symmetric boundary condition and (b) asymmetric boundary condition.

Fig. 3. The critical load–orientation angle of nanotubes relation of a carbon nanotube reinforced polyethylene matrix composite plate. The volume fraction is 10% for armchair carbon nanotubes (6, 6). (a) Symmetric boundary condition and (b) asymmetric boundary condition.
Fig. 4. The critical load– aspect ratio relation of a carbon nanotube reinforced polyethylene matrix composite plate. The orientation angle of nanotubes and the volume fraction is $\theta = 45^\circ$ and 10%, respectively, for armchair carbon nanotubes (6, 6). (a) Symmetric boundary condition and (b) asymmetric boundary condition

References


