EXACT BUCKLING SOLUTION OF COMPOSITE WEB/FLANGE ASSEMBLY

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1 Introduction

Composite materials are increasingly being used in aerospace structures and current design methodologies need to be improved in order to achieve lightweight efficient structures at low cost. The orthotropic properties of composite materials complexify analytical formulation development, especially for instability problems such as buckling. Buckling needs to be considered when designing a number of aerospace structures; for instance, it is an important design criteria for frames used to reinforce fuselage skins. In such a construction, the frames are attached to the fuselage skin and the junction between the frames and the skin provides some stiffness that can help prevent buckling. The web of the frames can be assumed to be a composite plate with a composite flange at one edge. Neglecting the radius of curvature, the frames buckling problem becomes one of a composite plate with a free flange and some boundary conditions on the remaining edges (Fig.1).

Many authors worked on buckling of composite plates with a combination of free, simply supported, clamped and rotationally restrained boundary conditions [1-7] and a few authors proposed a formulation for plates reinforced by a flange. Mittelstedt [8] worked on the case of a composite plate having three simply supported edges and one edge reinforced by a free flange. He presented an exact analytical formulation and some approximate formulations for the prediction of the buckling load that was in good agreement with results of finite element analysis. In the present work, we develop an analytical formulation for the local and lateral buckling analyses of a composite plate, representing the web of a frame reinforced with a flange, assuming a different set of boundary conditions (see Fig.1). In effect, Mittelstedt used a simply supported boundary condition at the edge representing the junction with the fuselage skin. In the present study, this edge is assumed to be clamped, as it is shown that having a simply supported boundary condition is too conservative. A comparison between the predictions of the analytical formulation and those of finite element analyses is performed, in order to validate the developed analytical model.

2 Problem statement

The considered structure is an orthotropic web, referred to as orthotropic plate on Fig. 1, of length $a$ and width $b$. The web is simply supported and loaded in compression ($\sigma_x$) at $x = 0$ and $x = a$ and clamped at $y = 0$. A free flange of width $h$ his considered at $y = b$, as shown on Fig.1. This flange consists of an orthotropic laminate and is represented by a extensional stiffness ($EA_f$), a bending stiffness ($EI_f$) and a torsional stiffness ($GJ_f$). The stress-strain relationship is based on classical laminate plate theory [9, 10] and symmetric and balanced stacking sequences are assumed. Thus, the bending-torsion coupling of the plate is neglected. The web thickness is assumed to be small compared to its dimensions and the displacements are linear and small compared to the thickness of the web. The following expression describes the relationship between the bending moments $M$ and the laminate curvature $\kappa^0$.
\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{66}
\end{bmatrix}
\begin{bmatrix}
\kappa_x^0 \\
\kappa_y^0 \\
\kappa_{xy}^0
\end{bmatrix}
\] (1)

The matrix coefficients \(D_{ij}\) (i, j = 1, 2 and 6) are the bending stiffness coefficients of the laminate. From the classical laminate theory, a relationship between curvature and deformation is given by:

\[
\kappa_x^0 = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y^0 = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy}^0 = -2 \frac{\partial^2 w}{\partial x \partial y}
\] (2)

where \(w\) is the out of plane displacement of the web. The web governing differential equation is:

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} = 0
\] (3)

The two loaded edges being simply supported at \(x = 0\) and \(x = a\), the deformation shape in the \(x\) direction is assumed to be sinusoidal:

\[
w(x, y) = \sin \left(\frac{mn}{a}\right) \varphi(y)
\] (4)

where \(\varphi(y)\) is assumed to be a function of \(y\) only and \(m\) expresses the number of half-waves in the buckling mode. Finding the fundamental solution for \(\varphi(y)\) [11], the deformation equation becomes:

\[
w(x, y) = \sin \left(\frac{mn}{a}\right) \left[B_1 \cosh \left(\frac{k_1 y m n}{a}\right) + B_2 \sinh \left(\frac{k_1 y m n}{a}\right) + B_3 \cos \left(\frac{k_2 y m n}{a}\right) + B_4 \sin \left(\frac{k_2 y m n}{a}\right)\right]
\] (5)

where \(B_{1,2,3,4}\) are unknown parameters and \(k_1\) and \(k_2\) are calculated [11]:

\[
k_{1,2} = \frac{1}{D_{22}} \sqrt{(D_{12} + 2D_{66})^2 + D_{22} \left(N_x \left(\frac{a}{mn}\right)^2 - D_{11}\right) \pm (D_{12} + 2D_{66})}
\] (6)

### 2.1 Boundary conditions

A clamped boundary condition is considered at \(y = 0\):

\[w|_{y=0} = 0\] (7)

\[\frac{\partial w}{\partial y}|_{y=0} = 0\] (8)

These two equations reduce the number of unknown parameters \(B_{1,2,3,4}\) from four to two:

\[B_3 = -B_4\] (9)

\[B_4 = -B_2 \frac{k_1}{k_2}\] (10)

The deformation shape equation is then reduced to:

\[w(x, y) = \sin \left(\frac{mn}{a}\right) \left[B_1 \left(cosh \left(\frac{k_1 y m n}{a}\right) - \cos \left(\frac{k_2 y m n}{a}\right)\right) + B_2 \left(hinh \left(\frac{k_1 y m n}{a}\right) - \frac{k_1}{k_2} \sin \left(\frac{k_2 y m n}{a}\right)\right)\right]\] (11)

At \(y = b\), the bending moment of the web works against the torsion of the flange, as stipulated by Mittelstedt [10]:

\[D_{12} \frac{\partial^2 w}{\partial x^2} + D_{22} \frac{\partial^2 w}{\partial y^2} - GF_{\delta} \frac{\partial^3 w}{\partial x^2 \partial y}|_{y=b} = 0\] (12)

Finally, a relationship between shear force in the web and the bending and compression in the flange exists at \(y = b\):

\[El_f \frac{\partial^4 w}{\partial x^4} + \delta N_x b \frac{\partial^2 w}{\partial x^2} - D_{22} \frac{\partial^3 w}{\partial y^3} - (D_{12} + 4D_{66}) \frac{\partial^3 w}{\partial x^2 \partial y}|_{y=b} = 0\] (13)

where \(N_x\) is the load per unit width on the web and \(\delta\) is the ratio of the flange extensional stiffness \((EA_f)\) to the web extensional stiffness \((EA_w)\):

\[\delta = \frac{EA_f}{EA_w} = \frac{EA_f}{E_i b t}\] (14)

where \(E_i\) is the equivalent Young’s modulus of the web in the \(x\) direction and \(t\) is the thickness of the web.
2.2 Solution

Equations (12) and (13) cannot be solved directly. Instead, the system of two equations is solved by setting its determinant equal to zero:

\[
V_1 \cosh \left( k_1 b \frac{\pi m}{a} \right) \cos \left( k_2 b \frac{\pi m}{a} \right) + V_2 \sinh \left( k_1 b \frac{\pi m}{a} \right) \cos \left( k_2 b \frac{\pi m}{a} \right) + V_3 \sin \left( k_1 b \frac{\pi m}{a} \right) \sin \left( k_2 b \frac{\pi m}{a} \right) + V_4 \sin \left( k_1 b \frac{\pi m}{a} \right) \sin \left( k_2 b \frac{\pi m}{a} \right) + V_5 = 0 \quad (15a)
\]

where

\[
V_1 = -2G_{Jf} \left( E_{If} \frac{\pi m}{a} \right)^2 - N_x b d 
- 2D_{12}(D_{12} + 4D_{66}) + 2D_{22}k_1^2k_2^2 
+ 2D_{22}(k_1^2 - k_2^2) \sinh \left( k_1 b \frac{\pi m}{a} \right) \cos \left( k_2 b \frac{\pi m}{a} \right) \quad (15b)
\]

\[
V_2 = D_{22}(k_1^2 + k_2^2) \left[ G_{Jf} \left( \frac{\pi m}{a} \right) k_2 + \frac{1}{k_1} \left( E_{If} \frac{\pi m}{a} \right)^2 - N_y b d \right] \quad (15c)
\]

\[
V_3 = D_{22}(k_1^2 + k_2^2) \left[ G_{Jf} \left( \frac{\pi m}{a} \right) k_1 - \frac{1}{k_1} \left( E_{If} \frac{\pi m}{a} \right)^2 - N_y b d \right] \quad (15d)
\]

\[
V_4 = -\frac{k_2}{2k_1} \left[ V_1 \left( k_1^2 \right)^2 - 1 \right] - 4D_{66}D_{22} \left( k_1^2 + k_2^2 \right) 
+ k_2^2 \left( \left( \frac{k_1}{k_2} \right)^2 + 1 \right) \quad (15e)
\]

\[
V_5 = D_{22}(k_1^2 + k_2^2)^2 - V_1 \quad (15f)
\]

Solving equation (15) for \( N_i \) is done using a numeric algorithm because of the transcendental nature of this equation.

3 Results

The formulation just developed for a clamped boundary condition at \( y = 0 \) is compared to that of a simply supported boundary condition using a simple example representing a typical fuselage frame. The frame web has a length \( a = 450 \) mm and a height \( b = 85 \) mm. The width of the flange \( h \) is ranged from 0 mm to 100 mm. The web stacking sequence is \( [(45^\circ/90^\circ/-45^\circ/0^\circ)]_S \) leading to a total thickness of 3.48 mm. The flange stacking sequence is \( [(45^\circ/90^\circ/-45^\circ/0^\circ)]_S \) corresponding to a total thickness of 5.51 mm. The flange and web are made of the same fibre reinforced material having the properties:

\[
E_{11} = 149 \text{ GPa} \\
E_{22} = 10.1 \text{ GPa} \\
v = 0.33 \\
G_{12} = 4.55 \text{ GPa} \\
\text{Ply thickness} = 0.145 \text{ mm}
\]

Buckling loads were predicted using the analytical formulation presented in the current work for the case of a clamped boundary condition (at \( y = 0 \)), and compared to those of Mittlestedt for the case of a simply supported boundary condition (at \( y = 0 \)). The analytical results were compared to the finite element analysis method using Nastran software. In the finite element model (Fig.2), the plates, both web and flange, are made of CQUAD4 elements having 4 nodes with 6 degrees of freedom (x, y, z translations and x, y, z rotations). A convergence study was conducted in order to refine the mesh to satisfactory results. For the size of the plates considered in this paper, the final mesh consists of square element of 5 mm width. Forces are applied on nodes of the web and flange edges at \( x = 0 \) and \( x = b \) (Fig.2). The first buckling mode obtained for each flange width was considered and the corresponding buckling load was recorded. Comparison of buckling loads predicted by the analytical formulation and finite element model are presented on Fig.3 for simply supported (Mittlestedt) and clamped boundary conditions at \( y = 0 \). Analytical formulation seems to be in good agreement with the finite element analysis for a flange width below 70 mm (clamped boundary condition) and 78 mm (simply supported boundary condition). With larger flanges, the buckling mode is changed from web buckling (local or lateral) to a flange local buckling mode (Fig.3), which is not captured in the analytical formulation. This flange local buckling mode must be studied using a different formulation as the analytical formulation presented in this study focuses on web buckling only. The results also demonstrate that, in this case, having a flange larger than 42 mm gives no additional stiffness to the frame against buckling. This can be seen as a minimal yet acceptable flange width [8] providing simply supported condition to the web.
The selected example shows that using a simply supported rather than a clamped boundary condition (at $y = 0$) reduces the predicted buckling load by 30% for large flanges and even more for small flanges. Assuming the real-life condition is closer to the clamped boundary condition, the simply supported boundary condition would lead to overdesign of the composite frame.

With a real airplane fuselage, the skin-stringer assembly gives a support to the frame web at the skin location ($y = 0$). The support that the skin-stringer assembly provides to the web is somewhere between the simply supported and clamped boundary conditions. To compare the simply supported and clamped boundary conditions with a realistic case, a skin-stringer support example is presented. This example represents a skin-stringer assembly having a
typical stiffness used in the aerospace industry. The stringer consists of an I-beam and is assumed to be perfectly fused to the skin. Dimensions and properties of the skin-stringer assembly are presented in Table 1. The finite element models used to find the rotational stiffness of the skin-stringer assembly (at $y = z = 0$) are presented in Fig.4 and Fig.5. In those models, a transverse load is applied on the frame to create a moment at $y = z = 0$ and the deformation angle is recorded. As shown on Fig.6, the average angle (neglecting edge effect) obtained with the skin-stringer assembly is matched using rotational springs. Then, those rotational springs are added to the frame finite element model (Fig.2) at $y = 0$. Those rotational springs are a third boundary condition for the web. Then, all three boundary conditions are compared using finite element analysis and results are shown on Fig.7. The results confirmed the previous assumption that the buckling load of a typical skin-stringer assembly is much closer to the buckling load of a clamped boundary condition than that of a simply supported boundary condition.

Fig.4 - Finite element model (skin, stringer and frame web) used to find the skin-stringer rotational stiffness at $y = 0$, displacements are indicated by $U$ and rotation by $R$. 
Fig. 5 - Finite element model (frame web and rotational springs) used to find the skin-stringer rotational stiffness at $y = 0$, displacements are indicated by $U$ and rotation by $R$.

Fig. 6 - Rotation at $y = 0$ using the skin-stringer and rotational springs finite element results.
Fig. 7 - Predicted buckling load for various boundary conditions at $y = 0$ using the finite element method. Boundary conditions are indicated by SS (simply supported), C (clamped) and RS (rotational springs representing the skin-stringer support). The properties of the skin-stringer assembly are indicated in Table 1.

Table 1 - Dimensions and laminates of the skin-stringer assembly

<table>
<thead>
<tr>
<th>Skin-Stringer</th>
<th>Laminate</th>
<th>Thickness</th>
<th>Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner flange</td>
<td>$[(45^\circ/90^\circ/-45^\circ/0^\circ)(45^\circ/90^\circ/-45^\circ/0^\circ)_s]$</td>
<td>3.48</td>
<td>3</td>
</tr>
<tr>
<td>Web</td>
<td>$[(45^\circ/90^\circ/-45^\circ/0^\circ)_s]$</td>
<td>1.45</td>
<td>20</td>
</tr>
<tr>
<td>Bottom flange</td>
<td>$[(45^\circ/90^\circ/-45^\circ/0^\circ)_s]$</td>
<td>1.45</td>
<td>10</td>
</tr>
<tr>
<td>skin</td>
<td>$[(45^\circ/90^\circ/-45^\circ/0^\circ)_s]$</td>
<td>1.16</td>
<td>150</td>
</tr>
</tbody>
</table>
4 Conclusion and future work

The present work demonstrated that assuming a simply supported boundary condition at the skin is a conservative assumption. On the other hand, the proposed formulation for a clamped boundary condition is closer to the reality but overestimates the buckling loads. Therefore, a mathematical formulation for the case of a rotationally restraining boundary condition will be developed in the future. This case would better represent the real-life boundary condition, which is somewhere between the simply supported and clamped boundary conditions. This will allow the development of efficient lightweight frames.

The formulation presented in the current paper is not a closed-form solution and needs to be solved numerically. In order to provide the aerospace industry with a quick way of estimating the buckling load of a frame, an approximate closed-form formulation should be developed in the future. Furthermore, the straight web-flange formulation presented in this study should be extended to incorporate the radius effect in order to get a more accurate representation of a real aerospace frame.

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References