1 Introduction and Objective

Unsymmetric laminations of composite plates can make the plates have a bi-stability due to the thermally-induced residual stress during the curing process [1]. The residual stress induces a curvature of the composite plates and it can be kept without any holding force because the flat state is unstable and the curved one is stable. The bi-stable plates can be used as various types of actuators for changing the shape of structures. In literature, multistable composite plates have also been studied for realizing adaptive structures [2].

The snap-through of unsymmetric fiber-reinforced composite laminates has been investigated by Dano and Hyer [3], where they use shape memory alloy (SMA) wires to induce the snap-through of composite panels. Further, piezoelectric patches called macro-fiber composite (MFC) [4] have been studied in order to actuate bi-stable structures [5, 6]. These studies have focused on static actuations where the actuation force is generated by smart materials such as SMA and lead zirconium titanate (PZT). Further, the MFCs are easily glued on the surface of flat and/or curved structure using suitable adhesives because they are flexible.

In order to further investigate a possibility of the sophisticated actuation methodology that possibly overcomes some remaining challenges of the morphing control of the bi-stable structures, some researchers have focused on dynamic behaviors of the bi-stable plates for applying the improvement of the snap-through control for morphing structures [7, 8].

More recently, Senba et al. [9] experimentally showed the snap-through of a [0/90] and [0/45] unsymmetric laminate can be induced by the vibration generated by a MFC. On the other hand, Arrieta et al. [10] studied a different configuration of composite laminates with multiple MFCs, in which a half of the plate is [0/90] and the other is symmetric [0₂] laminates. Their work showed that both snap-through and snap-back of the laminate were realized by changing the excitation frequency for the MFCs.

Although dynamically induced bi-stable phenomenon has been experimentally and theoretically shown [7-12] in several studies, few studies show that what is a significant parameter for the dynamic snap-through and how they are properly designed. In particular, Senba et al. described that additional mass (see Fig. 3) attached on the bi-stable laminate that increased an inertia force was needed to successfully induce dynamic snap-through [9, 10] whereas Arrieta et al. and Diaconu et al. indicated the dynamic snap-through without any additional mass [11, 12] was really observed. Thus, one cannot always say that dynamic snap-through is advantageous in practical applications especially for lightweight aerospace structures if such additional mass is necessary in a specific design of the unsymmetric laminates.

Therefore, to obtain a fundamental design strategy of bi-stable unsymmetric laminates that use vibration to make snap-through, this study investigates dynamic properties of a linear 1-D.O.F system. The system approximately expresses the vibration characteristics of the first vibration mode of the unsymmetric laminate that we have achieved the dynamically induced snap-through. This work is an extension of our previous study [12]. Finally, we formulate a design strategy especially to minimize the additional mass or input voltage for the MFCs.
2 Modeling for Dynamics of Bi-stable Unsymmetric Laminate

The Rayleigh method is used to derive the equation of motion for the equivalent 1-D.O.F linear system that approximately expresses the dynamic characteristics of the unsymmetric laminate. As a result, the equation of motion for the first vibration mode can be expressed by Eq. (1).

\[(m + \Delta m)x + c\ddot{x} + kx = f\]

where, \(x\), \(m\), \(c\), \(k\), and \(f\) are displacement, constant mass, damping, stiffness for the equivalent 1-D.O.F system, and external force. In addition, \(\Delta m\) is a contribution of point masses on the laminate, as shown in Fig. 3. Here, we assume that the harmonic excitation as \(f = F_0 e^{j\omega t}\) is applied to the system as Eq. (1) and the solution is written as \(x = A e^{j\omega t}\). Then, the frequency response amplitude under the harmonic excitation can be derived. Since the amplitude of excitation force can be assumed to be proportional to the input voltage for MFC, \(F\) can be written as \(F = aV\). When the parameters, \(m\), \(c\), and \(k\) are normalized by the coefficient \(a\), (e.g., \(\tilde{m} = m/a\)), then, the maximum value of the amplitude is given by Eq. (2).

\[|A|_{\text{max}} = \frac{2(\tilde{m} + \Delta \tilde{m})}{\sqrt{4(\tilde{m} + \Delta \tilde{m})\tilde{k} - \tilde{c}^2}} V\]

Note that Eq. (2) is valid only when \(2\tilde{m}\tilde{k} - \tilde{c}^2 > 0\). As shown in Eq. (2), the peak amplitude of vibration can be thought as a function of the additional mass, \(\Delta \tilde{m}\). Then, we experimentally identified the system by using the strain data for sinusoidal excitation with MFC actuator.

3 Experiments

In this experiment, the MFC actuation is used to induce the dynamic snap-through of bi-stable composite plates, which are unsymmetric laminates, [0/90/0\(^{MFC}\)]. The dimension and layup of the laminate is shown in Fig. 1. There are three layers: cross-ply laminate with two layers and MFC layer. Properties of each layer are shown in Table 1. The fiber direction of the MFC is the x-direction. Note that the laminate includes a 10mm margin for being clamped in vibration experiments; the width is slightly longer than the length of the plate.

3-1 Experimental Setup

The experimental setup is shown in Fig. 2. It consists of a signal generator (KENWOOD, FG-281), voltage amplifier (MESS-TEK, M-2601), strain gage (Tokyo Sokki Kenkyujo, GFLA-6-50), bridge box, dynamic strain meter (Tokyo Sokki Kenkyujo, DA-37A) and oscilloscope (YOKOGAWA, DL708E).

The strain gage is used to measure the x-axis strain as shown in Fig. 1, and strain is zero when the plate is in the state A without any clamping (i.e, one stable state without any constraint at the boundary). Hence, the strains in the state A after clamping in the cases 1 and 2 are not zero.

The signal generator is used to input sinusoidal signal to the high voltage amplifier. The output data of the strain gage are measured in order to identify the simplified model for the laminate.

The two stable states of the laminates are shown in Fig. 3, where the one end of the laminate is clamped. The strain gage is attached when the laminate is in the state A.

Table 1: Layer properties and dimensions for the unsymmetric laminates.

<table>
<thead>
<tr>
<th>Property</th>
<th>PYROFIL® #380</th>
<th>MFC (MP8557P1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_1), GPa</td>
<td>141</td>
<td>30.3</td>
</tr>
<tr>
<td>(E_2), GPa</td>
<td>10</td>
<td>15.9</td>
</tr>
<tr>
<td>Thickness, m</td>
<td>250x10(^{-6})</td>
<td>300x10(^{-6})</td>
</tr>
<tr>
<td>(L_x), m</td>
<td>0.148</td>
<td>0.111 (0.085)</td>
</tr>
<tr>
<td>(L_y), m</td>
<td>0.148</td>
<td>0.066 (0.057)</td>
</tr>
</tbody>
</table>
Fig. 1: Lamination and macro fiber composite (MFC) combination for $[0/90/0^{\text{MFC}}]$ composite laminate.

Fig. 2: Experimental setup for dynamic buckling of an unsymmetric laminate in the state A for the case 1.
3-2 Identification of a Linear Model

Figure 3 is photographs of the two stable states for the [0/90/0\_MFC] unsymmetric laminate, which was made with UD prepregs in our laboratory. The width and the length of the plate are 0.148 [m] and 0.138 [m] as described. First, we got strain data under sinusoidal excitation given by the MFC, where input voltage was 270[V] for the case 1 and 290[V] for the case 2. These input voltages were determined by another experiment. The relationship between the strain amplitude and some different excitation frequencies are shown in Table 2 for the case 1 and Table 3 for the case 2. Note that the circles in the third row of each table indicate that snap-through during the excitation occurred.

Then, the mass, stiffness, and damping parameters were identified with enough number of data. The identification method for these parameters has been explained in our previous study [12]. The method is based on the strain-based system identification approach [13].

The frequency responses of the unsymmetric laminate for the cases 1 and 2 were obtained by substituting the identified parameters into Eq. (2). The results are shown in Fig. 4. In Fig. 4, the estimated curve was plotted with actual data. The values of the identified parameters for the two cases are shown in Table 4.
3-3 Contribution of Additional Mass on Vibration Property

The total mass of additional masses for the case 1 is 0.06 [kg] while that for the case 2 is 0.116 [kg]. We analyzed the contribution of these additional masses on the vibration properties by applying the Rayleigh method. Although linear modal properties such as mode shape and modal frequency have been derived in recent study [14] by Grant and Hyer, the mode shape for the cantilevered unsymmetric laminates with pre-stress due to clamping has not yet been analyzed. In this study, the modal mass corresponding to the mass of the laminate and that to the additional masses are calculated separately, where the mode shape for the first vibration mode of the cantilevered unsymmetric laminate was approximated by a static displacement of the cantilevered beam under a concentrated load at the free end for a simplicity as,

$$\phi_1 = \frac{P x^2}{6EI} (3l_e - x)$$  \hspace{1cm} (3)

where, $P$, $l$, $EI$, and $x$ are the concentrated load, length of the beam, the bending stiffness of the beam, and $x$ coordinate, respectively. The origin of the $x$ coordinate is defined at clamped end of the plate.

![Diagram](image)

**Table 2:** Maximum strain before dynamic snap-through for the case 1.

<table>
<thead>
<tr>
<th>Excitation frequency (Hz)</th>
<th>5.4</th>
<th>5.5</th>
<th>5.6</th>
<th>5.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>strain ($\times 10^{-5}$)</td>
<td>195</td>
<td>240</td>
<td>240</td>
<td>200</td>
</tr>
<tr>
<td>snap-through</td>
<td>NA</td>
<td>0</td>
<td>0</td>
<td>NA</td>
</tr>
</tbody>
</table>

**Table 3:** Maximum strain before dynamic snap-through for the case 2.

<table>
<thead>
<tr>
<th>Excitation frequency (Hz)</th>
<th>4.6</th>
<th>4.7</th>
<th>4.8</th>
<th>4.9</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>strain ($\times 10^{-5}$)</td>
<td>260</td>
<td>340</td>
<td>340</td>
<td>330</td>
<td>318</td>
</tr>
<tr>
<td>snap-through</td>
<td>NA</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>NA</td>
</tr>
</tbody>
</table>

**Table 4:** Identified strain-based modal parameters for the equivalent 1-DOF system of the first vibration mode of the unsymmetric laminate for the cases 1 and 2.

<table>
<thead>
<tr>
<th>case</th>
<th>$\hat{m} + \Delta \hat{m}$</th>
<th>$\hat{c}$</th>
<th>$\hat{k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>case 1</td>
<td>$1.25e+4$</td>
<td>$3.12e+4$</td>
<td>$1.53e+7$</td>
</tr>
<tr>
<td>case 2</td>
<td>$7.71e+3$</td>
<td>$2.68e+4$</td>
<td>$7.20e+6$</td>
</tr>
</tbody>
</table>

Fig. 4: Frequency response amplitude of strain for the stable state B of the unsymmetric laminate.
By using Eq. (3), the modal masses for the case 1 can be calculated as,

\[ m = \frac{1}{2} \int_0^{x_1} \rho A \phi_1^2 dx = 0.0018 \left( \frac{P_0^2}{3EI} \right) \]  

(4)

\[ \Delta m = \frac{1}{2} \Delta M_1 \phi_1^2(x_1) = 0.0124 \left( \frac{P_0^2}{3EI} \right) \]  

(5)

where, \( \rho \), \( A \), and \( x_1 \) are density of the laminate, cross sectional area of the laminate, and x-coordinate where the additional mass for the case 1 is attached to the laminate. The density of the laminate is assumed to be 1500 [kg/m\(^3\)]. The x-coordinate \( x_1 \) was approximated as the average position of the additional mass used in the experiments [12], \( x_1 = 0.0335 \) [m].

The results we obtained by using Eqs. (4) and (5) indicated that the contribution of the point masses was estimated by simply evaluating the ratio as,

\[ \frac{\Delta \hat{m}}{\hat{m} + \Delta \hat{m}} = \frac{\Delta m}{m + \Delta m} \approx 0.87 \]  

(6)

where, the first equality is derived because the external force \( F \) is constant. Thus, approximately 87% of the total mass, \( \hat{m} + \Delta \hat{m} \) is a contribution of the additional mass. In other word, when the additional mass is zero, the peak amplitude, which can be calculated using Eq. (2), decreases to 36% of the original one and required voltage for the snap-through of unsymmetric laminates increases accordingly.

The modal mass for the case 2 is also calculated using Eq. (3). Since the plate itself is same as that used for the case 1, only the contribution of additional masses for the case 2 is calculated as,

\[ \Delta \hat{m} = \frac{1}{2} \Delta M_2 \phi_1^2(x_2) = 0.0293 \left( \frac{P_0^2}{3EI} \right) \]  

(7)

where, \( x_2 \) is the x-coordinate where the additional mass for the case 2 is attached to the laminate. The x-coordinate of the additional mass for the case 2 was approximated as the average position of the additional masses, \( x_2 = 0.027 \) [m]. As a result, the contribution of the point masses for the case 2 becomes approximately 94% of the total mass, \( \hat{m} + \Delta \hat{m} \). Thus, the contribution of the additional mass for the case 2 is about 10% greater than that for the case 1.

Although the contribution of the additional mass on the vibration amplitude given by Eq. (2) is evaluated based on the displacement-based analysis, the experiments described later uses only the strain data. Thereby, we translate the analytical results obtained above as Eqs. (4-7). The relationship between strain-based modal mass and displacement-based modal mass is expressed by

\[ \hat{m} = \phi^T m_0 = \phi^T \phi \hat{m} \]  

(8)

where, \( \phi \) is a vector to transform displacement-based mass to strain-based one [13]. Since this paper considers only the first mode and the modal mass is a constant, the second equality is satisfied. Therefore, following equality is obtained as,

\[ \frac{\Delta \hat{m}}{\hat{m} + \Delta \hat{m}} = \frac{\Delta \hat{m}}{\hat{m} + \Delta \hat{m}} \]  

(9)

where, \( \hat{m}, \Delta \hat{m} \) are strain-based modal masses. Thus, the contribution of the additional mass on the strain-based modal mass becomes equal to that derived from displacement-based modal mass.

3-4 Design Strategy for Bi-stable Composite Laminate

Based on the discussions above, we formulate a fundamental design strategy for the bi-stable composite laminates induced by vibration. The design problem such as optimization of a lamination of composite plate is not considered. Instead, we will concentrate how to determine the additional mass or input voltage for MFC.

Two example design problems are considered: 1) to minimize the additional mass attached to the laminate when available input voltage for the MFC actuator is given; 2) to minimize the input voltage when additional mass is given. Each problem can be solved as following procedures.

Assume that vibration test for the nominal value of the additional mass \( \Delta M_0 \) is performed. Then, a following equation is assumed to be satisfied,
where \( \tilde{m}_0, \tilde{c}_0, \tilde{k}_0, \Delta \tilde{m}_1 \) are modal mass, damping, stiffness for the initial input voltage \( V_0 \), and modal mass for the additional mass, respectively. \( V_1 \) is a required input voltage when the additional mass \( \Delta M_1 \) is attached to the plate. \( |A_0| \) is a maximum amplitude for the bi-stable plate with nominal additional mass \( \Delta M_0 \), which is also equal to the critical amplitude that is necessary for the snap-through of the bi-stable plate.

For the first design problem, the required condition for the additional mass can be derived by solving Eq. (10) about \( \Delta \tilde{m}_1 \). As a result,

\[
\Delta \tilde{m}_1 \geq \frac{-\beta + \sqrt{\beta^2 - 4\alpha \gamma}}{2\alpha}
\]

where, \( \alpha, \beta, \gamma \) are defined as follows:

\[
\alpha = 4V_1^2, \quad \beta = 8\tilde{m}_0V_1^2 - 4|A_0|^2\tilde{c}_0^2\tilde{k}_0
\]

\[
\gamma = 4\tilde{m}_0V_1^2 - |A_0|^2\tilde{c}_0^2(4\tilde{m}_0\tilde{k}_0 - \tilde{c}_0^2)
\]

Once the modal mass for the additional mass is calculated, then additional mass is easily obtained since the modal mass is proportional to the additional mass as Eq. (5) or (7).

On the other hand, for the second design problem, the required condition for the input voltage can be derived by solving Eq. (10) about \( V_1 \) as,

\[
V_1 \geq \frac{c_0\sqrt{4(\tilde{m}_0 + \Delta \tilde{m}_1)\tilde{k}_0 - \tilde{c}_0^2}}{2(\tilde{m}_0 + \Delta \tilde{m}_1)}|A_0|
\]

Equations (11) and (12) can be selected by users according to the design requirements of the unsymmetric laminates.

The design procedures can be logically formulated. Here, only the first design problem is described since the second problem is also easily treated by the same manner as the first one. The design process can be divided into the following steps:

**Step 1**: Perform vibration tests using unsymmetric laminates with the MFC and nominal additional mass \( \Delta M_0 \) to find the minimum input voltage, \( V_0 \) and critical strain amplitude \( |A_0| \).

**Step 2**: Identify modal parameters (see section 3-2).

**Step 3**: Calculate the contribution of additional mass for plate by Eq. (4) and for the additional mass by Eq. (5).

**Step 4**: Calculate the right hand side of Eq. (11) using identified modal parameters and \( |A_0| \).

**Step 5**: Determine the additional mass \( \Delta M_1 \) using Eq. (11).

Note that the total of the additional masses used in the experiment was 60 [g] for the case 1 and 116 [g] for the case 2; they are greater than the mass of the unsymmetric laminate itself (i.e., 15g). However, the additional mass can be minimized by considering allowable input voltage for the MFC. Also, to reduce the input voltage, decreasing the damping ratio of the unsymmetric laminate is effective. Section 3-5 presents example results for these design problems.

It is noted that the assumed mode shape as Eq. (3) for the 1-D.O.F model can be improved by Rayleigh-Ritz method proposed Ref. [14].

### 3-5 Required Input Voltage vs. Additional Mass

Figure 5(a) shows the relationship between the required voltage and additional mass which was calculated by Eq. (12) for the dynamic snap-through from the state B to the state A. The experimental data is also plotted. The required voltage depends on the additional mass because it increases the peak amplitude for the resonance for the state B. It is also shown that when the damping coefficient \( c \) in Eq. (1) decreases 50% of the original one (see broken line in Fig. 5), the additional mass can be drastically decreased. As Eq. (2) indicates, the change in damping coefficient is more effective than the variation of the mass. For example, when the applied voltage is about 400 [V], the broken line in Fig. 5 shows that the additional mass is no longer needed to make snap-through during vibration.

On the other hand, Fig. 5 (b) shows the results for the case 2. The required voltage without additional mass was greater than that for the case 1. In addition, it is shown that if the additional mass for the case 2
is the same as the case 1, the required voltage becomes about 400 [V], which is much greater than that required for the case 1 (i.e, 270[V]).

![Graph of required voltage vs. additional mass for two cases.](image)

(a) Case 1 (Exp.: $V_0=270$ [V] and 0.06 [kg]).

(b) Case 2 (Exp.: $V_0=290$ [V] and 0.116 [kg]).

Fig. 5: Required voltage for dynamic snap-through vs. additional mass for the two cases.

4 Conclusions

This study investigated the effect of the additional mass on the dynamical behavior of bi-stable composite laminates in order to obtain a design strategy of bi-stable unsymmetric laminates induced by vibration. A linear 1-D.O.F system was used to analyze the contribution of the additional mass on the reduction of required input voltage for the piezoelectric actuator attached to the bi-stable laminates. As a result, it is found that the additional mass attached to the unsymmetric laminates was necessary to increase of the resonance amplitude. Further, the results showed that if the damping coefficient of the laminate is lower, the required voltage can be effectively reduced.

References


