THE 19TH INTERNATIONAL CONFERENCE ON COMPOSITE MATERIALS

STRESS ANALYSIS FOR PURE BENDING OF COMPOSITE TUBES WITH ARBITRARY WINDING ANGLES INCLUDING 0° OR 90°

C. Zhang¹, S. V. Hoa², P. Liu¹

¹ Department of Civil Engineering, Xiamen University, Xiamen, 361005 China
² Concordia Center for Composites, Department of Mechanical and Industrial Engineering, Concordia University, Montreal, H3G1M8, Canada

* Corresponding author (chzhang@xmu.edu.cn)

Abstract: A method is proposed to analyze the bending of composite tubes made up of layers of special angle 0° or 90° together with other layers of arbitrary angle. It is found that in an earlier approach some of the parameters are singular for a few layer orientations even though the stresses as well as the displacements they described are nonsingular. The investigation of the parameters near special winding angles indicates that the singular parameters are inversely proportional to the winding angle increment. So they tend to infinite when winding angle tends to the special value. In conventional technique, the terms with singular parameters should be eliminated since the stresses and displacements are finite. However, for the special layers together with the ordinary layers in this paper, such terms with singular parameter cannot be simply eliminated. Otherwise, the continuity conditions on the interface between special and ordinary layers cannot be satisfied. Based on this knowledge, new unified coefficients as well as their corresponding parameters are introduced and an efficient approach is suggested which is nonsingular for any case with layer angle 0° or 90° and others. The results for several tubes are provided to illustrate the proposed method. The numerical analysis by NASTRAN are also employed for comparison. They are in good agreement to each other.

Keywords: stress analysis; composite tube; pure bending; nonsingular parameter; cylindrically orthotropic layer; arbitrary winding angles

1 Introduction

The composite tube consisting of an assembly of several coaxial circular hollow cylinders has been proved to be very useful for many structures. A large number of methods have been developed to calculate the structures in composite tubes. Lekhnitskii [1] obtained the system of partial differential equations for the case of cylindrical anisotropy. He has worked out the simpler case of one cylinder subjected to axisymmetric load. He has also provided the solution for one cylindrically orthotropic cylinder under axial force and bending moment, where he derived the stress field including unknowns firstly and then used the free boundary conditions and load conditions to obtain the available equations to determine these unknowns. This solution for pure bending has also been obtained by other methods such as the state space method by Tarn and Wang [2], the finite strip method by Sun et al. [3]. In addition, the theoretical formulations are provided by Shadmehri, Derisi, and Hoa [4] to determine the equivalent flexural stiffness (EI) for composite tubes where the in-plane deformation is neglected. The Donnell theory is used by Fuchs and Hyer [5] to compute the linear response of the displacements and interlaminar stresses for the thin, symmetrically slaminated circular cylinders subject to bending by end rotations. However, we would like to point out that the solution provided by Lekhnitskii cannot be obtained by his proposed method [6], because the number of unknowns is larger than the number of the available equations. Jolicoeur and Cardou [7] extended the method by Lekhnitskii to a more general case of cylindrical anisotropy for a system including several coaxial hollow circular cylinders, with and without the core. The continuity conditions of stress and displacement on the interface between every layer and the boundary conditions of stress on the free surfaces are used to determine the unknown coefficients. Then the equivalent flexural stiffness (EI) can be obtained. Finally, the load condition is used to determine the curvature of bending. For the general case of cylindrical anisotropy, the solutions can be obtained successfully. However, for cylindrical orthotropy [0/90], there are two terms
with infinite parameters, thus the relating continuity conditions on the interface and boundary conditions on the free surfaces cannot be used straightforwardly. Even though Jolicoeur and Cardou [7] declared that their results are equivalent to those by Lekhnitskii [1], the detail was not provided. Zhang and Hoa [6] proposed the limit-based approach to overcome this problem. However, they only considered the case with two orthotropic cylinders [0/90] and without other arbitrary angles.

In this paper, the bending of tubes made up of layers with arbitrary winding angles including 0° or 90° and other angles is considered. The singular parameters near special angle are investigated. Then the nonsingular unified coefficients as well as their corresponding parameters are introduced and an efficient method is proposed.

2 Preliminary

A composite tube made of several coaxial circular layers of cylinder is shown in Fig.1. These layers are made of homogeneous orthotropic composite material with three principle directions. The angle between the fiber direction and the plane through the axis of cylinders, is called the winding angle. If the winding angle is 0° or 90°, the layer is cylindrically orthotropic. The tube consisting of cylindrically orthotropic layers, particularly with inner layers of 90°, is significant in engineering because their fibers can contribute to the stability around the circumference. The transformation of compliance elastic matrix from the material principle coordinate system to the cylindrical system can be readily obtained as follows,

\[ C = T^T S T \]

where

\[ T = \begin{bmatrix} 0 & s^2 & -2\tilde{s}\tilde{c} & 0 & 0 \\ 0 & \tilde{c}^2 & \tilde{s}^2 & 2\tilde{s}\tilde{c} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{s} & -\tilde{c} & 0 \\ 0 & 0 & 0 & \tilde{c} & -\tilde{s} & 0 \\ 0 & \tilde{s}\tilde{c} & -\tilde{s}\tilde{c} & \tilde{s}^2 - \tilde{c}^2 & 0 & 0 \end{bmatrix} \]  

in which \( s = \sin \phi, \tilde{c} = \cos \phi \), and

\[ S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \]

in which

\[ S_{11} = \frac{1}{E_{11}}, S_{22} = \frac{1}{E_{22}}, S_{33} = \frac{1}{E_{33}} \]

\[ S_{44} = \frac{1}{G_{12}}, S_{55} = \frac{1}{G_{13}}, S_{66} = \frac{1}{G_{12}} \]  

\[ S_{12} = \frac{-\nu_{12}}{E_{11}}, S_{13} = \frac{-\nu_{13}}{E_{11}}, S_{23} = \frac{-\nu_{23}}{E_{22}} \]

Suppose all layers including special angles \( \tilde{\phi} = 0° \) or \( \tilde{\phi} = 90° \) and others are bounded perfectly. Let angles of all these layers change together by an additional angle \( \delta \), namely, the changed special angles can be expressed as following

\[ \phi = \tilde{\phi} + \delta \]  

It is found that the transformation from \( \tilde{\phi} = 0° \) to \( \phi = 0° + \delta \) is similar to the transformation from \( \tilde{\phi} = 90° \) to \( \phi = 90° + \delta \) as follows

\[ C = T^T \tilde{C} T \]

where \( \tilde{C} \) is the compliance elastic matrix for \( \tilde{\phi} \) and

Fig. 1  Pure bending composite tube
STRESS ANALYSIS FOR PURE BENDING COMPOSITE TUBE WITH
ARBITRARY WINDING ANGLES INCLUDING 0° OR 90°

\[ T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & c^2 & s & 2sc & 0 & 0 \\
0 & s^2 & c^2 & -2sc & 0 & 0 \\
0 & -sc & sc & c^2 - s^2 & 0 & 0 \\
0 & 0 & 0 & 0 & c & -sc \\
0 & 0 & 0 & 0 & s & c \\
\end{bmatrix} \]  
(7)

in which \( s = \sin \delta, \ c = \cos \delta \). So we can consider these two cases in the similar procedures. In other words, the further discussion is available both for \( \bar{\delta} = 0^\circ \) and for \( \bar{\delta} = 90^\circ \). Expanding Eq. (6), the compliance elastic matrix is expressed in the following form in which the nonzero parameters are identified,

\[ C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\
C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\
C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\
C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & C_{56} \\
0 & 0 & 0 & 0 & C_{65} & C_{66} \\
\end{bmatrix} \]  
(8)

The reduced elastic parameters are also employed as

\[ \beta_j = C_{ij} \frac{C_{3j}}{C_{33}} \]  
(9)

Its relating matrix can be expressed in the following form as

\[ \beta = \begin{bmatrix}
\beta_{11} & \beta_{12} & 0 & 0 & 0 & 0 \\
\beta_{12} & \beta_{22} & 0 & 0 & 0 & 0 \\
\beta_{14} & \beta_{24} & 0 & 0 & 0 & 0 \\
\beta_{14} & \beta_{24} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \beta_{55} & \beta_{56} \\
0 & 0 & 0 & 0 & \beta_{56} & \beta_{66} \\
\end{bmatrix} \]  
(10)

By means of these reduced elastic parameters, four roots for the characteristic equation are given by

\[ m_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]  
(11)

in which \( a, b, c \) are the expressions of \( \beta_j \). Besides, the following parameters can also be obtained as

\[ \begin{align*}
\mu_1 &= \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}^{-1} \begin{bmatrix} 2c \end{bmatrix} \\
\mu_2 &= \begin{bmatrix} a_{21} & a_{22} \end{bmatrix} C_{33} \end{align*} \]  
(12)

where

\[ \begin{align*}
a_{11} &= -2\beta_{14} - 6\beta_{24} + \beta_{56} \\
a_{12} &= 4\beta_{44} - \beta_{55} \\
a_{21} &= -\beta_{11} - 2\beta_{12} + 3\beta_{22} - \beta_{66} \\
a_{22} &= \beta_{44} - 2\beta_{24} + \beta_{56} \end{align*} \]  
(13)

By means of Eqs. (10) and (11), the following parameters can be calculated for \( i = 1, 2, 3, 4 \)

\[ g_i = \frac{\beta_{24}m_i^2 + (\beta_{14} + \beta_{24})m_i - \beta_{56}}{\beta_{44}m_i^2 - \beta_{55}} \]  
(14)

For the problem of pure bending, the in-plane and out-of-plane stresses are expressed as

\[ \sigma_r = (\kappa_r \sin \theta - \kappa_s \cos \theta) \times \begin{bmatrix} \sum_{i=1}^{4} K_i r_i m_i^{n-1} + \mu_i r \end{bmatrix} \]  
(15)

\[ \sigma_\theta = (\kappa_r \sin \theta - \kappa_s \cos \theta) \times \begin{bmatrix} \sum_{i=1}^{4} K_i m_i + 1)r_i^{n-1} + 3\mu_i r \end{bmatrix} \]  
(16)

and

\[ \tau_{rz} = (\kappa_r \cos \theta + \kappa_s \sin \theta) \times \begin{bmatrix} \sum_{i=1}^{4} K_i g_i r_i m_i^{n-1} + \mu_r r \end{bmatrix} \]  
(17)

Thus the longitudinal stress can be calculated as

\[ \sigma_r = \frac{1}{C_{13}} (\kappa_r \sin \theta - \kappa_s \cos \theta) \times \begin{bmatrix} -C_{15} \sigma_r - C_{23} \sigma_o - C_{34} \tau_{oz} \end{bmatrix} \]  
In addition, the displacements can also be expressed using the parameters in Eqs. (11), (14), and (12). It is found that the parameters \( g_i \)'s are only included in the out-of-plane stresses in Eq. (16), and as such
in the longitudinal stress in Eq. (17), while they are included in all displacements.

3 Problems from singular parameters

For special layers when $\phi = \delta$, i.e., by Eq. (5), when $\delta = 0$, the compliance elastic matrix has the following form as

$$
\bar{C} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{C}_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{C}_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{C}_{66}
\end{bmatrix}
$$

(18)

Then, by Eq. (14) with Eqs. (19) and (22), one can calculate the following

$$
\bar{g}_i = 0 \text{ for } i = 1, 2
$$

(24)

However, by Eq. (14) with Eqs. (19) and (22), $\bar{g}_3$ and $\bar{g}_4$ cannot be straightforwardly worked out because their denominators and numerators are equal to zero. So, the description for the stresses and displacements by Jolicoeur and Cardou [7] is not satisfactory where $\bar{g}_3$ and $\bar{g}_4$ are included. It is our attempt in this paper to find a suitable description for them.

4 Investigation of parameters near special angles

4.1 Approximation of parameters

Because $\bar{g}_3$ and $\bar{g}_4$ cannot be determined, the approximations for $g_3$ and $g_4$ near $\delta = 0$ cannot be straightforwardly derived using their Taylor series expansion. So, we rewrite $\bar{g}_i$’s in the following form as

$$
\bar{g}_i = A_i \bar{B}_i \quad \text{for } i = 1, 2, 3, 4
$$

(25)

where

$$
\begin{align*}
A_i &= \beta_{24}^2 m_i^2 + (\beta_{14} + \beta_{24}) \bar{m}_i - \beta_{56} \\
B_i &= \beta_{44}^2 m_i^2 - \beta_{55}
\end{align*}
$$

(26)

It is found, for special layers $\delta = 0$, that

$$
\begin{align*}
\bar{A}_i &= 0 \text{ for } i = 1, 2, 3, 4 \\
\bar{B}_i &\neq 0 \text{ for } i = 1, 2 \\
\bar{B}_i &= 0 \text{ for } i = 3, 4
\end{align*}
$$

(27)

Since $A_i$’s and $B_j$’s are continuous functions of $\delta$, their approximations can be obtained using their Taylor series expansions. Thus, the approximations for $g_i$’s can be obtained. The following derivatives can be derived as

$$
\begin{align*}
\bar{A}_i' &= \beta_{24}^2 \bar{m}_i^2 + (\beta_{14} + \beta_{24}) \bar{m}_i - \beta_{56} \\
\bar{A}_i^* &= 0 \\
\bar{B}_i' &= 0 \\
\bar{B}_i^* &= \beta_{44}^2 \bar{m}_i^2 + \beta_{44} 2 \bar{m}_i \bar{m}_i^* - \beta_{55}
\end{align*}
$$

(28)

Consequently, the following approximations can be obtained as
STRESS ANALYSIS FOR PURE BENDING COMPOSITE TUBE WITH ARBITRARY WINDING ANGLES INCLUDING 0° OR 90°

\[ A_i = A_i \delta + \frac{1}{2!} A_i^2 \delta^2 + \cdots \equiv A_i \delta \]  \hspace{1cm} (29)

and

\[
\begin{align*}
B_i &= B_i \delta + \frac{1}{2!} B_i^2 \delta^2 + \cdots \equiv B_i \\
\text{for } i &= 1, 2, 3, 4 \\
B_i &= B_i \delta + \frac{1}{2} B_i^2 \delta^2 + \cdots \equiv \frac{1}{2} B_i^2 \delta^2 \\
\text{for } i &= 1, 2 \\
B_i &= B_i \delta + \frac{1}{2} B_i^2 \delta^2 + \cdots \equiv \frac{1}{2} B_i^2 \delta^2 \\
\text{for } i &= 3, 4
\end{align*}
\]  \hspace{1cm} (30)

Then, by Eq. (25) with Eqs. (29) and (30), one has

\[ g_i \equiv \frac{A_i}{B_i} \delta \text{ for } i = 1, 2 \]  \hspace{1cm} (31)

and

\[ g_i \equiv \frac{2A_i}{B_i^2} \frac{1}{\delta} \text{ for } i = 3, 4 \]  \hspace{1cm} (32)

This indicates that the approximations for \( g_1, g_2 \) are proportional with \( \delta \) while those for \( g_3, g_4 \) are inversely proportional with \( \delta \). In other words, \( g_1, g_2 \) are infinite for special layers as

\[ |g_i| = \infty \text{ for } i = 3, 4 \]  \hspace{1cm} (33)

where the symbols with wave ' represent their limits, i.e.,

\[ \tilde{g}_i = \lim_{\delta \to 0} g_i \]  \hspace{1cm} (34)

However, the stresses and displacements for special layers are finite before failure. That is to say, in the method by Jolicoeur and Cardou [7], the finite stresses and displacements for special layers are described by the infinite \( \tilde{g}_1 \) and \( \tilde{g}_2 \). So the further calculation is difficult.

4.2 Unavailable by previous technique

By previous technique, the terms with infinite parameters are eliminated according to the finite stresses and displacements. In other words, one simply sets

\[ \tilde{K}_i = 0 \text{ for } i = 3, 4 \]  \hspace{1cm} (35)

By Eq. (16) with Eq. (35) as well as Eqs. (22) and (24), the following can be obtained as

\[
\begin{align*}
\tau_{\theta \theta} &= 0 \\
\tau_{rz} &= 0
\end{align*}
\]  \hspace{1cm} (36)

However, it is found by the actual numerical analysis that the solution in Eq. (36) are only available for the tube with special layers and without ordinary layers [6]. In general, for the tube including special layers together with ordinary layers, this solution does not work. Otherwise, the continuity condition on the interface between special and ordinary layers cannot be satisfied. For example, for the special layer \( n + 1 \), the continuity condition for \( \tau_{rz} \) on their interface \( r = b_n \) requires

\[ (\tau_{rz}(b_n))_n = (\tau_{rz}(b_{n+1}))_{n+1} \]  \hspace{1cm} (37)

But the expression in Eq. (36) for total special layer \( n \) cannot provide such condition since for ordinary layer \( n + 1 \) it is possible

\[ (\tau_{rz}(b_{n+1}))_{n+1} \neq 0 \]  \hspace{1cm} (38)

So, for this case, the previous technique in Eq. (35) to deal with the terms with infinite parameters does not work. In fact, since the infinite singular parameters cannot be obtained in actual calculation, the method by Jolicoeur and Cardou [7] is difficult to be used for the special layers in engineering. Consequently, more investigation is necessary.

5 Investigation of stress coefficients for special layers

5.1 Limit to the expression for stresses

Since some of the singular parameters are infinite as discussed earlier, they are difficult to describe the finite stresses and displacements for special layers. To overcome this problem, we have to find their finite alternatives. For this purpose, we consider the stress limits when \( \delta \to 0 \). For those finite parameters, since they are continuous functions of \( \delta \), their limits are equal to their values. In addition, due to the fact that the stresses for special layers are finite, their limits are also equal to their values. For this knowledge, the limit to in-plane stresses and out-of-plane stresses for special layers can be expressed as

\[
\begin{align*}
\tau_{\theta \theta} &= 0 \\
\tau_{rz} &= 0
\end{align*}
\]
\[ \sigma_r = (\kappa_x \sin \theta - \kappa_y \cos \theta) \times \left( \sum_{i=1}^{4} \hat{K}_i r^{m_i-1} + \mu_i r \right) \]
\[ \sigma_\theta = (\kappa_x \sin \theta - \kappa_y \cos \theta) \times \left( \sum_{i=1}^{4} \hat{K}_i (\mu_i + 1) r^{m_i-1} + 3 \mu_i r \right) \]
\[ \bar{\sigma}_{\theta\theta} = -(\kappa_x \cos \theta + \kappa_y \sin \theta) \times \left( \sum_{i=1}^{4} \hat{K}_i r^{m_i-1} + \mu_i r \right) \]
\[ \bar{\sigma}_{r\theta} = -(\kappa_x \sin \theta - \kappa_y \cos \theta) \times \left( \sum_{i=1}^{4} \hat{K}_i g_i r^{m_i-1} \right) \]
\[ \bar{\sigma}_r = (\kappa_x \cos \theta + \kappa_y \sin \theta) \times \left( \sum_{i=1}^{4} \hat{K}_i g_i r^{m_i-1} \right) \]

(39)

It should be noted that in Eq. (40) we consider \( \mu_1 r = \mu_2 r = 0 \) according to \( \mu_1 = \mu_2 = 0 \). However, we cannot simply let \( \hat{K}_1 \hat{g}_1 = 0 \) and \( \hat{K}_2 \hat{g}_2 = 0 \) according to \( \hat{g}_1 = \hat{g}_1 = 0 \) and \( \hat{g}_2 = \hat{g}_2 = 0 \) because we do not have knowledge about \( \hat{K}_1 \) and \( \hat{K}_2 \). So we consider \( \hat{K}_1 \hat{g}_1 \) and \( \hat{K}_2 \hat{g}_2 \) together in case \( \hat{K}_1 \) and \( \hat{K}_2 \) are infinite. Of course, \( \hat{K}_1 \hat{g}_3 \) and \( \hat{K}_2 \hat{g}_4 \) have to be considered together because \( \hat{g}_3 \) and \( \hat{g}_4 \) are infinite in Eq. (33).

5.2 Requirement for finite in-plane stresses

Consider any four regions which are different to each other inside the special layer \( n \):
\[ r = r_i, \ 0 \leq \theta \leq 2\pi \quad \text{for} \ k = 1, 2, 3, 4 \]
(41)

where
\[ b_n < r_1 < r_2 < r_3 < r_4 < b_{n+1} \]
(42)
in which \( b_n \) and \( b_{n+1} \) are the inner and outer radii for the special layer \( n \). We can consider either \( \sigma_r \) or \( \sigma_\theta \) in Eq. (39). For the sake of convenience, they are rewritten in the following form as

\[ \sigma_r = (\kappa_x \sin \theta - \kappa_y \cos \theta) \bar{S}(r) \]
\[ \sigma_\theta = -(\kappa_x \cos \theta + \kappa_y \sin \theta) \bar{S}(r) \]

(43)

where

\[ \bar{S}_i(r) = \sum_{i=1}^{4} \hat{K}_i r^{m_i-1} + \mu_i r \]
(44)

Since both of them are finite for any \( \theta \in [0, 2\pi] \), they provide the following equivalent equations as

\[ \bar{S}_i(r_1) = \sum_{i=1}^{4} \hat{K}_i r_1^{m_i-1} \]
\[ \bar{S}_i(r_2) = \sum_{i=1}^{4} \hat{K}_i r_2^{m_i-1} \]
\[ \bar{S}_i(r_3) = \sum_{i=1}^{4} \hat{K}_i r_3^{m_i-1} \]
\[ \bar{S}_i(r_4) = \sum_{i=1}^{4} \hat{K}_i r_4^{m_i-1} \]

(45)

As \( r_1 \neq r_2 \neq r_3 \neq r_4 \) by Eq. (42), one has

\[ \begin{vmatrix} r_1^{m_1-1} & r_2^{m_1-1} & r_3^{m_1-1} & r_4^{m_1-1} \\ r_1^{m_2-1} & r_2^{m_2-1} & r_3^{m_2-1} & r_4^{m_2-1} \\ r_1^{m_3-1} & r_2^{m_3-1} & r_3^{m_3-1} & r_4^{m_3-1} \\ r_1^{m_4-1} & r_2^{m_4-1} & r_3^{m_4-1} & r_4^{m_4-1} \end{vmatrix} \neq 0 \]
(46)

Thus the four vectors on right side in Eq. (45) are independent to each other. In addition, all vectors on both sides in Eq. (43) are finite, so one can conclude that

\[ |\bar{K}_i| < \infty \quad \text{for} \ i = 1, 2, 3, 4 \]
(47)

In other words, the finite in-plane stresses require that all the stress coefficients should be finite. Moreover, because the stresses as well as all the parameters are continuous functions of \( \delta \), the following can also be obtained as

\[ \bar{K}_i = \bar{K}_i \quad \text{for} \ i = 1, 2, 3, 4 \]
(48)

By Eq. (47) and (48) as well as Eq. (24), we now can conclude that

\[ \bar{K}_i \hat{g}_i = \bar{K}_i \hat{g}_i = \bar{K}_i \hat{g}_i = 0 \quad \text{for} \ i = 1, 2 \]
(49)

It indicates that the out-of-plane stresses in Eq. (40) only include the terms with infinite singular \( g_3 \) and \( g_4 \). Also, only the out-of-plane stresses include these singular parameters. Thus, to find the satisfactory description for stresses and displacements for special cylinders, one can make use of the requirement for finite out-of-plane stresses.
5.3 Requirement for finiteness of out-of-plane stresses

Consider the special layer \( n \), the following continuity conditions should be satisfied

\[
\begin{align*}
\left[(\tau_{rz}(b_n))_n = (\tau_{rz}(b_{n+1}))_{n+1}\right] \\
\left[(\tau_{rz}(b_{n+1}))_n = (\tau_{rz}(b_{n+1}))_{n+1}\right]
\end{align*}
\]

By means of Eq. (50) with Eqs. (40) and (49), the following can be derived as

\[
\begin{align*}
\sum_{i=3}^{4} K_i g_i \frac{\partial}{\partial \theta} \left(b_n^{i-1} - b_{n+1}^{i-1}\right) = t_n^{i-1} \\
\sum_{i=3}^{4} K_i g_i \frac{\partial}{\partial \theta} \left(b_{n+1}^{i-1} - b_{n+1}^{i-1}\right) = t_{n+1}^{i-1}
\end{align*}
\]

where

\[
\begin{align*}
t_n^{i-1} &= \sum_{i=3}^{4} (K_i)_{n-1} (g_i)_{n-1} b_n^{(m)_{n-1}} + (\mu_i)_{n-1} b_n \\
t_{n+1}^{i-1} &= \sum_{i=3}^{4} (K_i)_{n+1} (g_i)_{n+1} b_n^{(m)_{n+1}} + (\mu_i)_{n+1} b_n
\end{align*}
\]

Due to

\[
\begin{align*}
b_n^{i-1} - b_{n+1}^{i-1}, b_n^{i-1} - b_{n+1}^{i-1} \neq 0
\end{align*}
\]

Thus

\[
\begin{align*}
\left[\begin{array}{c}
K_3 g_3 \\
\text{or } K_4 g_4
\end{array}\right] = \left[\begin{array}{cc}
b_n^{i-1} - b_{n+1}^{i-1} & b_n^{i-1} - b_{n+1}^{i-1} \\
\end{array}\right]^{-1} \left[\begin{array}{c}
t_n^{i-1} \\
t_{n+1}^{i-1}
\end{array}\right]
\end{align*}
\]

Clearly, the values of expressions in Eq. (52) are finite since all the stresses inside the tube are finite. Therefore the limits on left side in Eq. (54) should be finite, i.e.,

\[
K_i g_i < \infty \text{ for } i = 3, 4
\]

Due to the continuity about \( \delta \), the following can be obtained

\[
K_i g_i = \bar{K}_i g_i \text{ for } i = 3, 4
\]

On the other hand, because \( g_3 \) and \( g_4 \) are infinite near \( \delta = 0 \), \( K_3 \) and \( K_4 \) should be infinitesimal near \( \delta = 0 \). Thus their limits are equal to zero, i.e.,

\[
\bar{K}_i = 0 \text{ for } i = 3, 4
\]

Consider Eq. (48) with Eq. (57), one has

\[
\bar{K}_i = 0 \text{ for } i = 3, 4
\]

In other words, the finite out-of-plane stresses require \( \bar{K}_3 = 0 \) and \( \bar{K}_4 = 0 \). This conclusion is similar to Eq. (35). However, Eqs. (57) and (58) are derived systematically while Eq. (35) is obtained only by simply eliminating the terms with infinite parameters. Moreover, the further conclusion in this paper by Eq. (57), which will be discussed in the following text, is different from that by Eq. (35).

Discussion for the simple case: only including one special layer

In the case for the tube including only one special layer, both \( r = b_1 \) and \( r = b_2 \) are free surfaces. The boundary conditions for \( \tau_{rz} \) on these free surface can be expressed as

\[
\begin{align*}
\tau_{rz}(b_1) &= 0 \\
\tau_{rz}(b_2) &= 0
\end{align*}
\]

By means of Eq. (59) with Eqs. (40) and (49), one has

\[
\begin{align*}
\sum_{i=3}^{4} K_3 g_3 b_n^{i-1} = 0 \\
\sum_{i=3}^{4} K_4 g_4 b_n^{i-1} = 0
\end{align*}
\]

Thus the following can be obtained

\[
\bar{K}_i g_i = \bar{K}_i g_i = 0 \text{ for } i = 3, 4
\]

This implies that \( \bar{\tau}_{rz} = 0 \), \( \bar{\tau}_{rz} = 0 \) inside total special layer for \( b_1 \leq r \leq b_2 \), \( 0 \leq \theta \leq 2\pi \).

It can be interpreted for this simple case that \( K_3 \) and \( K_4 \) are infinitesimal more than first-order. In other words, they can be expanded near \( \delta = 0 \) by using Taylor series expansion where \( \bar{K}_3 = 0 \) and \( \bar{K}_4 = 0 \) and meanwhile \( \bar{K}_3' = 0 \) and \( \bar{K}_4' = 0 \). In other words, for the tube only including one special layer, not only \( \bar{K}_3 = 0 \) and \( \bar{K}_4 = 0 \), but also \( \bar{K}_3 g_3 = 0 \) and \( \bar{K}_4 g_4 = 0 \) due to \( \bar{K}_3' = 0 \) and \( \bar{K}_4' = 0 \). It should be pointed that this solution is equivalent to Lekhnitskii [1]. However, we derive this solution systematically while Lekhnitskii set his unwanted coefficient to zero without any explanation. In addition, it should be noted that, even though Jolicoeur and Cardou [7]...
declared that they also obtained the results equivalent to Lekhnitskii [1], they did not provide the details.

**Discussion for the general case: including special layers and others**

In the general case when there are ordinary layers together with the special layers included in the tube, in Eq. (50) it is possible for $r_{z_c} \neq 0$ on the interface between other layers, namely,

$$
\left\{ \begin{array}{l}
(r_{z_c}(b_{n}))_{n-1} \\
(r_{z_c}(b_{n+1}))_{n+1}
\end{array} \right\} \neq \left\{ \begin{array}{l}
0 \\
0
\end{array} \right\}
$$

(62)

So there exists nonzero solution for Eq. (51), namely,

$$
\left\{ \begin{array}{l}
\bar{K}_3g_3 \\
\bar{K}_4g_4
\end{array} \right\} \neq \left\{ \begin{array}{l}
0 \\
0
\end{array} \right\}
$$

(63)

This indicates that it is possible for $\bar{K}_3g_3 \neq 0$ or $\bar{K}_4g_4 \neq 0$ even though the relating limit $\bar{K}_3 = 0$ and $\bar{K}_4 = 0$ (for this case we would refer to their limits than their values). In other words, it is possible for $r_{z_c} \neq 0$ inside the special layers for $b_n \leq r \leq b_{n+1}$, $0 \leq \theta \leq 2\pi$. Therefore, the terms with $\tilde{g}_3$ and $\tilde{g}_4$ cannot be simply eliminated by the previous technique.

In this case $K_3$ and $K_4$ can be considered as the infinitesimal of first-order. In other words, they can be expanded near $\delta = 0$ by using Taylor series expansion where $\tilde{K}_3 = 0$ and $\tilde{K}_4 = 0$ while $\bar{K}_3 \neq 0$ or $\bar{K}_4 \neq 0$. In other words, for the tube including special layers and together with other layers, one has $\tilde{K}_3 = 0$ and $\tilde{K}_4 = 0$ but $\bar{K}_3g_3 \neq 0$ or $\bar{K}_4g_4 \neq 0$ due to $\bar{K}_3 \neq 0$ or $\bar{K}_4 \neq 0$. This is why it is possible for $r_{z_c} \neq 0$ for special layers while $\bar{K}_3 = 0$ and $\bar{K}_4 = 0$.

For this reason, $r_{z_c}$ and $r_{\theta_c}$ inside the special cylinders would rather be expressed by $\bar{K}_3g_3$ and $\bar{K}_4g_4$ than by $\tilde{K}_3$ and $\tilde{K}_4$ together with $\tilde{g}_3$ and $\tilde{g}_4$.

**6 New method for analysis of composite tube including special layers and others**

Based on the previous discussion, the unified coefficients can be introduced as

$$
K_i^* \equiv \begin{cases}
K_i g_i, \quad \varphi \neq \tilde{\varphi} \\
\bar{K}_i, \quad \varphi = \tilde{\varphi}
\end{cases}
$$

for $i = 3, 4$ (64)

Clearly, such unified coefficients are finite for any case of $\varphi \in [0, 2\pi]$. Furthermore, the corresponding parameters can also be defined. From Eq. (64) one has

$$
K_i = K_i^* g_i^{-1} \quad \text{for } i = 3, 4 \quad \text{when } \varphi \neq \tilde{\varphi}
$$

(65)

Thus, consider Eq. (25), the new parameters for $\varphi \neq \tilde{\varphi}$ can be introduced as

$$
g_i^* \equiv g_i^{-1} = \frac{B_i}{A_i} \quad \text{for } i = 3, 4
$$

(66)

It is noted that such parameters cannot be calculated directly for special layer when $\varphi = \tilde{\varphi}$ because $\tilde{A}_3 = \tilde{A}_4 = 0$ and $\tilde{B}_3 = \tilde{B}_4 = 0$ in Eq. (27). However, according to Eq. (32), their limits are finite, i.e.,

$$
g_i^* = \lim_{\delta \to 0} g_i^* = \lim_{\delta \to 0} \frac{B_i}{2A_i} \delta = 0 \quad \text{for } i = 3, 4
$$

(67)

Therefore, the parameters particularly for special cylinders can be defined as

$$
g_i^* = 0 \quad \text{for } i = 3, 4
$$

(68)

The new parameters in Eqs. (66) and (68) can be rewritten together as

$$
g_i^* = \begin{cases}
g_i^{-1} \quad \text{for } \varphi \neq \tilde{\varphi} \\
0 \quad \text{for } \varphi = \tilde{\varphi}
\end{cases}
$$

for $i = 3, 4$ (69)

Such parameters are finite in any case of $\varphi \in [0, 2\pi]$. By means of the finite unified coefficients and their corresponding parameters in Eqs. (64) and (69), respectively, the stresses and displacements can be described satisfactorily. The in-plane stresses can be expressed as
STRESS ANALYSIS FOR PURE BENDING COMPOSITE TUBE WITH ARBITRARY WINDING ANGLES INCLUDING 0° OR 90°

\[
\sigma_r = (\kappa_r \sin \theta - \kappa_r \cos \theta) \times \\
\left( \sum_{i=1}^{2} K_i r^{m_i-1} + \sum_{i=3}^{4} K_i^* r^{m_i-1} + \mu_i r \right) \\
\sigma_\theta = (\kappa_\theta \sin \theta - \kappa_\theta \cos \theta) \times \\
\left( \sum_{i=1}^{2} K_i (m_i + 1) r^{m_i-1} + \sum_{i=3}^{4} K_i^* (m_i + 1) r^{m_i-1} + 3 \mu_i r \right) \\
\tau_{r\theta} = -(\kappa_r \cos \theta + \kappa_\theta \sin \theta) \times \\
\left( \sum_{i=1}^{2} K_i g_i r^{m_i-1} + \sum_{i=3}^{4} K_i^* g_i r^{m_i-1} + \mu_i r \right) \\
\text{while the out-of-plane stresses can be expressed as}
\]

\[
\tau_{rz} = (\kappa_r \cos \theta + \kappa_\theta \sin \theta) \times \\
\left( \sum_{i=1}^{2} K_i g_i r^{m_i-1} + \sum_{i=3}^{4} K_i^* g_i r^{m_i-1} + \mu_i r \right) \\
\tau_{\theta z} = -(\kappa_r \sin \theta - \kappa_\theta \cos \theta) \times \\
\left( \sum_{i=1}^{2} K_i g_i m_i r^{m_i-1} + \sum_{i=3}^{4} K_i^* m_i r^{m_i-1} - 2 \mu_i r \right)
\]

By Eqs. (70) and (71) the longitudinal stress can be expressed in the same form as Eq. (17). In addition, the in-plane and out-of-plane displacements can be expressed as follows,

\[
u = (\kappa_r \sin \theta - \kappa_\theta \cos \theta) \left( \frac{z^2}{2} + \sum_{i=1}^{2} K_i U_i' r^{m_i} + \sum_{i=3}^{4} K_i^* U_i' r^{m_i} \right) + \sum_{i=1}^{2} U_i' r^{m_i} + U_i' r^2 + \nu \]

\[
u = (\kappa_r \cos \theta + \kappa_\theta \sin \theta) \left( \frac{z^2}{2} + \sum_{i=1}^{2} K_i V_i' r^{m_i} + \sum_{i=3}^{4} K_i^* V_i' r^{m_i} \right) + \sum_{i=1}^{2} V_i' r^{m_i} + V_i' r^2 + \nu \]

and

\[
w = zr(\kappa_r \sin \theta - \kappa_\theta \cos \theta) + \\
(\kappa_r \cos \theta + \kappa_\theta \sin \theta) \times \\
\left( \sum_{i=1}^{2} K_i W_i' r^{m_i} + \sum_{i=3}^{4} K_i^* W_i' r^{m_i} + W_i' r^2 \right)
\]

where \( \nu \) is the curvature of rigid displacement and

\[
U_i' = \frac{1}{m_i} (\beta_1 + \beta_2 (m_i + 1) - \beta_{i4} g_{i4}) \\
V_i' = \frac{1}{m_i} (\beta_1 + \beta_2) \\
W_i' = \frac{1}{m_i} (\beta_{i5} g_{i5} - \beta_{56} g_{56}^*) \\
\text{for } i = 1, 2 \\
U_i' = \frac{1}{m_i} ((\beta_1 + \beta_2 (m_i + 1)) g_{i4}^* - \beta_{i4} m_i) \\
V_i' = \frac{1}{m_i} ((\beta_1 + \beta_2 - \beta_{i2} m_i) g_{i4}^* - m_{i2} (\beta_{i4} - \beta_{i2} m_i)) \\
W_i' = \frac{1}{m_i} (\beta_{i5} - \beta_{56} g_{56}^*) \\
\text{for } i = 3, 4 \\
U_i' = \frac{1}{2} (\mu_i (\beta_{11} + 3 \beta_{12}) - 2 \beta_{i4} \mu_{i4}) + \frac{C_{13} - 2 C_{33}}{2} \\
V_i' = \frac{1}{2} (\mu_i (\beta_{11} + \beta_{12} - 6 \beta_{22}) - 2 \mu_{i2} (\beta_{i4} - 2 \beta_{24}) + \frac{C_{13} - 2 C_{33}}{2} \\
W_i' = \frac{1}{2} (\beta_{i5} \mu_i - \beta_{56} \mu_{i4})
\]

By means of our description for stresses and displacements, the further procedures can be followed similar to Jolicoeur and Cardou [7] as:

Step 1: The unknown \((K_1)_n\), \((K_2)_n\), \((K_3)_n\), \((K_4)_n\) are determined by using the boundary conditions on the free surfaces of the tube and the continuity conditions on the interface between every layers.

Step 2: The end moment \(M_x, M_y\) are used to determine the curvatures for tube

\[
(\text{EI}) = \sum_{n=1}^{\infty} (\text{EI})_n
\]

where
\[(EI)_n = \frac{\pi}{(C_{33})_n} \left( \sum_{i=1}^{2} (K_i)_n((C_{13})_n + \right.
(C_{23})_n((m_1)_n + 1) - b^{-2}_{m+1} + \left. \sum_{i=1}^{4} (K_i^*)_n(g_i^*)_n((C_{13})_n + \right. (C_{23})_n((m_2)_n + 1) - b^{-2}_{m} + \left. (\mu)_n((C_{13})_n + 3(C_{23})_n - 1) \right) \right) \frac{b^d - b^i}{4} \] (76)

Step 3: The results for stresses as well as displacements for tube are calculated by Eqs. (70), (71), (72), and (73).

It should be pointed that we do not work out \(K_i\)'s and \(g_i\)'s or \(K_i^*\)'s and \(g_i^*\)'s to derive \(K_3^*\) and \(K_4^*\) by using Eq. (64). In fact \(K_3^*\) and \(K_4^*\) are determined by using the boundary and continuity conditions with \(g_3^*\) and \(g_4^*\) which are defined in Eq. (69).

7 Numerical examples
The stress analysis of five composite tubes which are made up of cylindrically anisotropic layers of different winding angles subject to pure bending is presented. The material parameters are provided in Table 1 while the geometry parameters for all tubes are provided in Table 2. To provide the pure bending moment at the end of tubes, we consider the linearly distributed load which is equivalent to the moment as

\[M = 1.5102 \times 10^4 \text{Nm}\]

The present approach is introduced to work out the solution. The bending stiffness (EI) for the tubes are provided in Table 3. The stress distributions are also shown in Figures 2 to 5.

The numerical analysis by Nastran for the same

<table>
<thead>
<tr>
<th>Tube</th>
<th>Ply sequence</th>
<th>Length /m</th>
<th>Radii for cross section /m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[90]</td>
<td>0.6</td>
<td>0.025</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>[90/0]</td>
<td>0.6</td>
<td>0.032</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>[90/45]</td>
<td>0.6</td>
<td>0.025</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>[90/45/0]</td>
<td>0.6</td>
<td>0.035</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>[90/±45/0]</td>
<td>0.6</td>
<td>0.039</td>
</tr>
</tbody>
</table>

Table 1  Graphite/polymer composite

<table>
<thead>
<tr>
<th>Material property</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_{11})</td>
<td>155</td>
<td>GPa</td>
</tr>
<tr>
<td>(E_{22})</td>
<td>12.1</td>
<td>GPa</td>
</tr>
<tr>
<td>(E_{33})</td>
<td>12.1</td>
<td>GPa</td>
</tr>
<tr>
<td>(G_{23})</td>
<td>3.2</td>
<td>GPa</td>
</tr>
<tr>
<td>(G_{13})</td>
<td>4.4</td>
<td>GPa</td>
</tr>
<tr>
<td>(G_{12})</td>
<td>4.4</td>
<td>GPa</td>
</tr>
<tr>
<td>(\nu_{23})</td>
<td>0.458</td>
<td>-</td>
</tr>
<tr>
<td>(\nu_{13})</td>
<td>0.248</td>
<td>-</td>
</tr>
<tr>
<td>(\nu_{12})</td>
<td>0.248</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2  Geometry model for tubes

<table>
<thead>
<tr>
<th>Tube</th>
<th>Ply sequence</th>
<th>Length /m</th>
<th>Radii for cross section /m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[90]</td>
<td>0.6</td>
<td>0.025</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>[90/0]</td>
<td>0.6</td>
<td>0.032</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>[90/45]</td>
<td>0.6</td>
<td>0.025</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>[90/45/0]</td>
<td>0.6</td>
<td>0.035</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>[90/±45/0]</td>
<td>0.6</td>
<td>0.039</td>
</tr>
</tbody>
</table>

Table 3  Results for tubes

<table>
<thead>
<tr>
<th>Tube</th>
<th>(\varphi)</th>
<th>(EI)\times10^4 \text{Nm}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>90°</td>
<td>18.641</td>
</tr>
<tr>
<td>2</td>
<td>0°</td>
<td>154.33</td>
</tr>
<tr>
<td>3</td>
<td>45°</td>
<td>17.166</td>
</tr>
<tr>
<td>4</td>
<td>45°</td>
<td>12.153</td>
</tr>
<tr>
<td>5</td>
<td>0°</td>
<td>87.980</td>
</tr>
</tbody>
</table>

The numerical analysis by Nastran for the same
STRESS ANALYSIS FOR PURE BENDING COMPOSITE TUBE WITH ARBITRARY WINDING ANGLES INCLUDING 0° OR 90°

tubes is also employed to check the present approach. The finite element models for all tubes are provided in Table 4. The results by Nastran in Figures 2 to 5 are shown in points of different types. It is found that the results by present approach and Nastran are in good agreement. So the present approach can be used to obtain the satisfactory solution for the pure bending composite tubes made of layers of cylindrical orthotropy with arbitrary angles including the special angles.

From the results, it is found that $\tau_{rz} = 0$ for Tube 1 [90] and Tube 2 [90/0] where only the special layers are included, while $\tau_{rz} \neq 0$ for Tube 3 [90/45] and Tube 4 [90/45/0] because the special layers are adjacent to the layer 45°. However, in Tube 5 [90/±45/0] $\tau_{rz}$ are almost zero even though the special layers are adjacent to other layers. This is because the other layers are wound in opposite angles, i.e., in 45° and -45°.

8 Conclusions

A method is proposed to overcome the problem from singular parameters for pure bending composite tube when the winding angles include 0° or 90°. The Taylor series expansion is employed to derive the approximations for the parameters near special winding angles. Then the unified coefficients as well as their corresponding parameters are introduced and an efficient method is proposed. Good agreement of the results for several composite tubes by the proposed approach and NASTRAN indicates that the approach is effective.

<table>
<thead>
<tr>
<th>Tube</th>
<th>Element type</th>
<th>Element number</th>
<th>Node number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Hex8</td>
<td>64000</td>
<td>68900</td>
</tr>
<tr>
<td>2</td>
<td>Hex8</td>
<td>64000</td>
<td>68900</td>
</tr>
<tr>
<td>3</td>
<td>Hex8</td>
<td>64000</td>
<td>68900</td>
</tr>
<tr>
<td>4</td>
<td>Hex8</td>
<td>96000</td>
<td>102950</td>
</tr>
<tr>
<td>5</td>
<td>Hex8</td>
<td>128000</td>
<td>136160</td>
</tr>
</tbody>
</table>
References


